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KNO-G scaling and lognormal shape of multiplicity distributions

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Multiplicity distributions - notation

$\sqrt{s}$ – center of mass energy of e^+e^-, pp or any XX collision

n – variable = 0, 1, 2, 3, …

n_{ch} – number of charge particles created in the collision

- e^+e^-, $p\dot{p}_{ann.}$, $\underline{\nu}$p: $n_{ch} = 0+2n = 0, 2, 4, 6, \ldots$
- pn, νn, $\underline{\nu}$n: $n_{ch} = 1+2n = 1, 3, 5, 7, \ldots$
- pp, $p\dot{p}_{nona.}$, $\pi^{\pm}$p, $K^{\pm}$p, νp: $n_{ch} = 2+2n = 2, 4, 6, 8, \ldots$

P_n – probability to observe n

$\langle n \rangle$ – average n

D_k – k-th dispersion of the distribution P_n

$$D_k = \langle (n-\langle n \rangle)^k \rangle^{1/k}$$

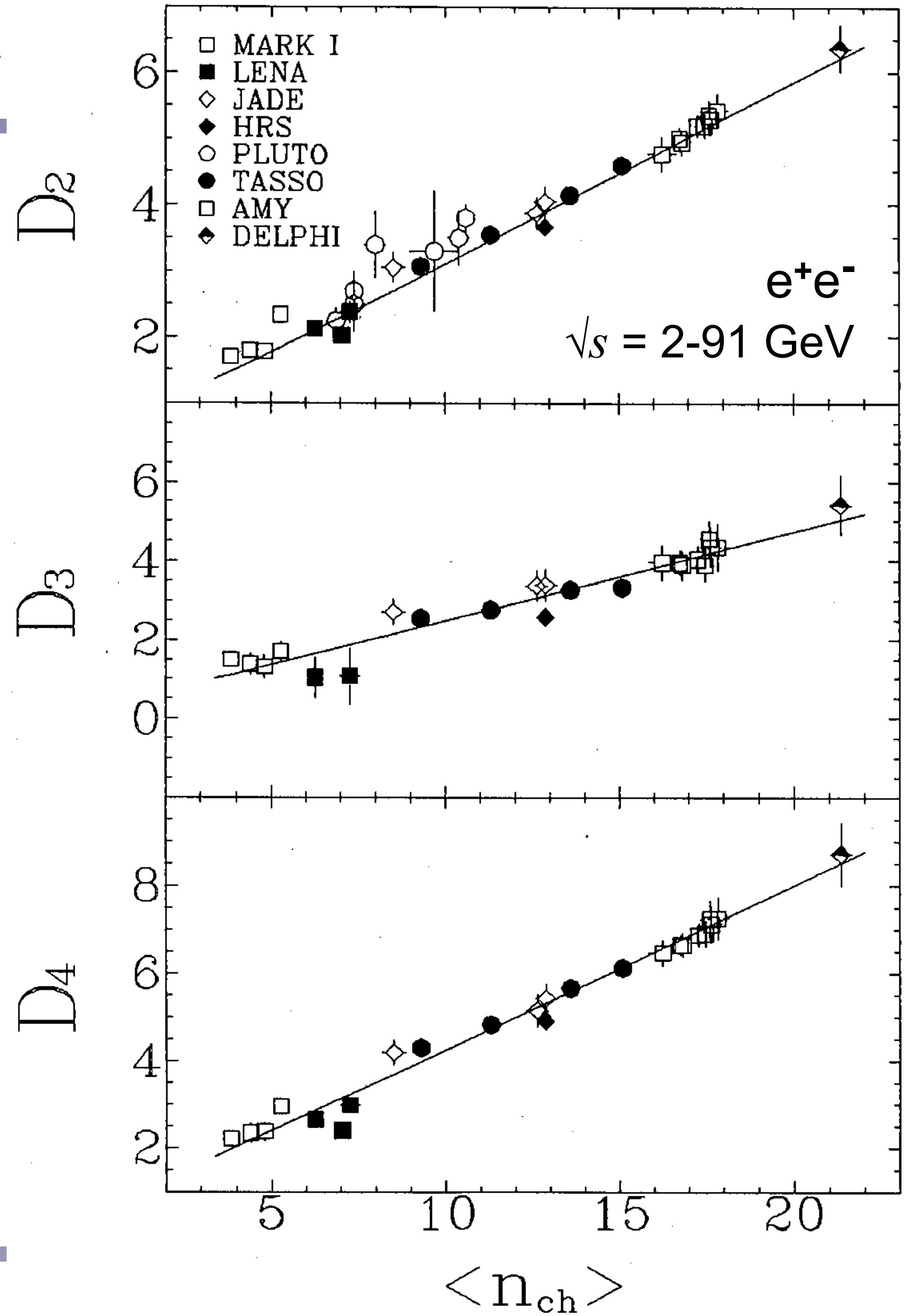
Basic observations

Wróblewski law [1]:

$$D_2 \sim \langle n \rangle$$

Important hint for dynamics, because one would naively expect „grenade explosion" with poissonian distribution of n and

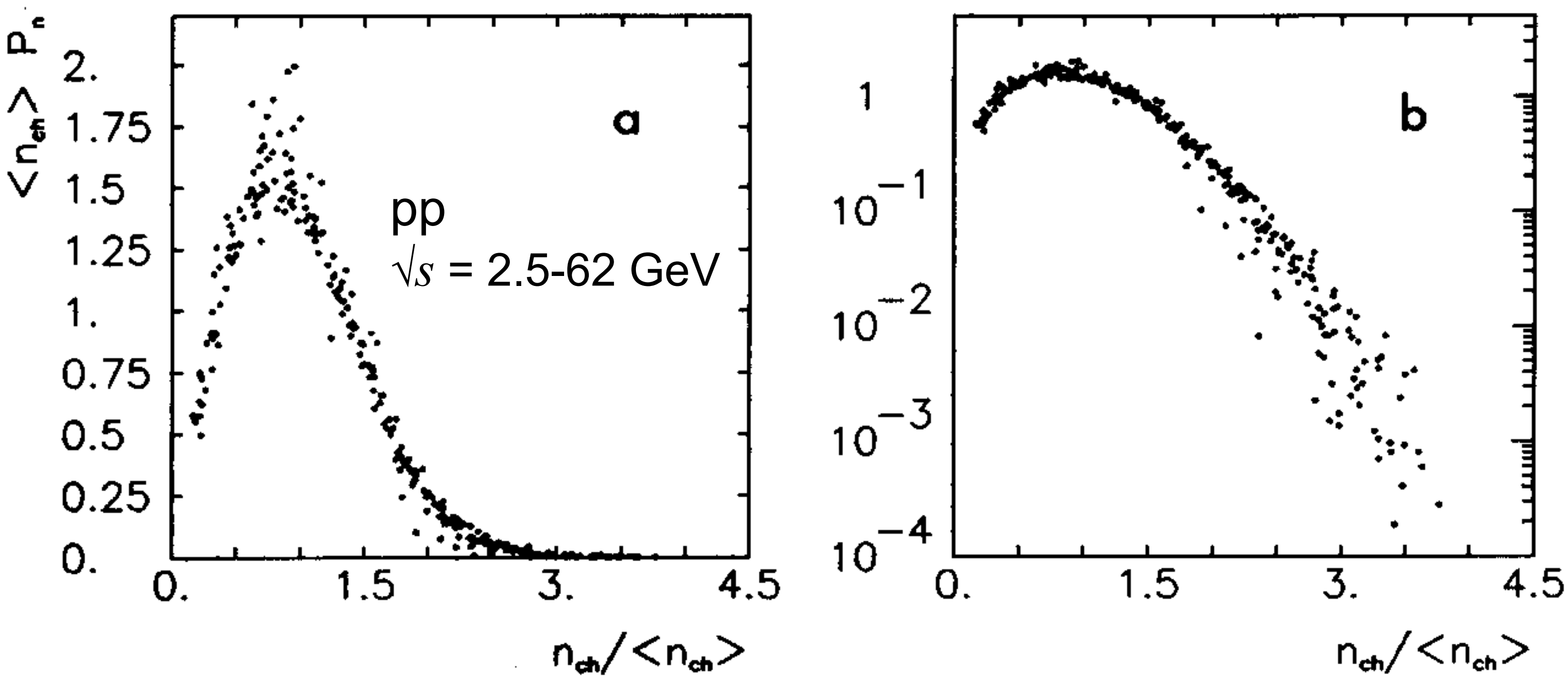
$$D_2 \sim \sqrt{\langle n \rangle}$$



KNO-scaling

Koba, Nielsen & Olesen observed [2], that P_n for different $\sqrt{s}$ can be described by a single scaling function ψ_{KNO}

$$P_n = \frac{1}{\langle n \rangle} \psi_{KNO} \left(\frac{n}{\langle n \rangle} \right)$$

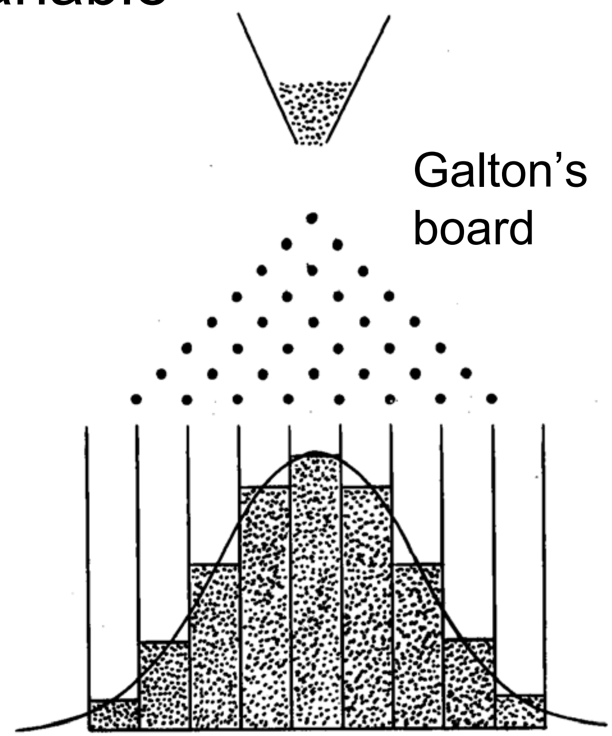


KNO-G scaling

- Golokhvastov [3] improved KNO scaling for low $\langle n \rangle$
- He replaced approximate scaling of P_n by exact scaling of its density function $f(\tilde{n})$
  - tilda denotes that $\tilde{n}$ is countinuous variable

$$P_n = \int_n^{n+1} f(\tilde{n}) d\tilde{n}$$

$$f(\tilde{n}) = \frac{1}{\langle \tilde{n} \rangle} \psi \left(\frac{\tilde{n}}{\langle \tilde{n} \rangle} \right)$$



Galton's board

KNO-G scaling

$$P_n = \int\limits_{n/\langle\tilde{n}\rangle}^{(n+1)/\langle\tilde{n}\rangle} \psi(z)dz \ , \ z = \frac{n}{\langle\tilde{n}\rangle} \ , \ \langle\tilde{n}\rangle = \int\limits_0^\infty \tilde{n}\psi(z)dz$$

It is convinient to use, instead of $\psi(z)$,

its primitive function $\phi(z)$

$$\phi(z) = -\int\limits_z^\infty \psi(z)dz$$

$$P_n = \phi\left(\frac{n+1}{\langle\tilde{n}\rangle}\right) - \phi\left(\frac{n}{\langle\tilde{n}\rangle}\right)$$

Useful approximations

$\langle n \rangle \approx \langle \tilde{n} \rangle - 0.5$

$D \sim \langle \tilde{n} \rangle$

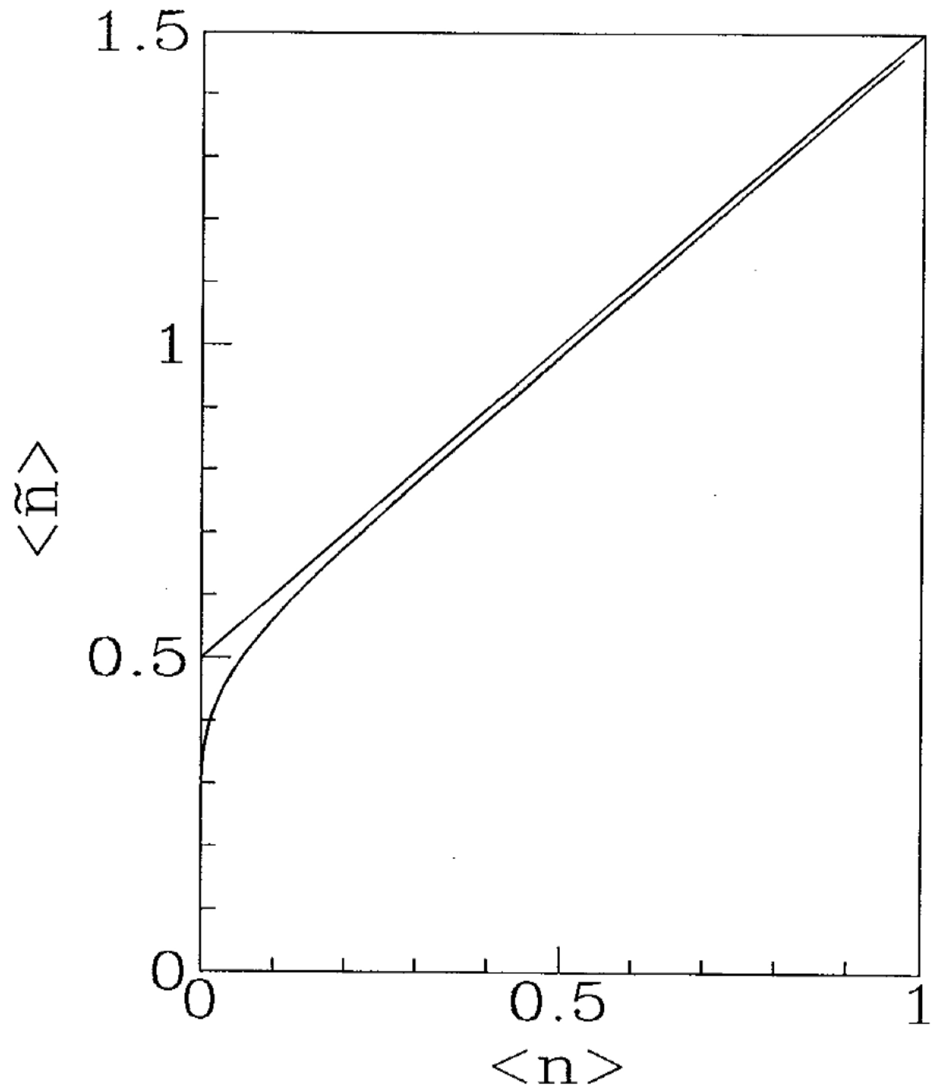
$D \sim (\langle n \rangle + 0.5)$

$\langle n_{ch}{}^{e+e-} \rangle \approx 2\langle \tilde{n} \rangle - 1$

$D_{ch}{}^{e+e-} \sim (\langle n_{ch}{}^{e+e-} \rangle + 1)$

$\langle n_{ch}{}^{pp} \rangle \approx 2\langle \tilde{n} \rangle - 1 + 2$

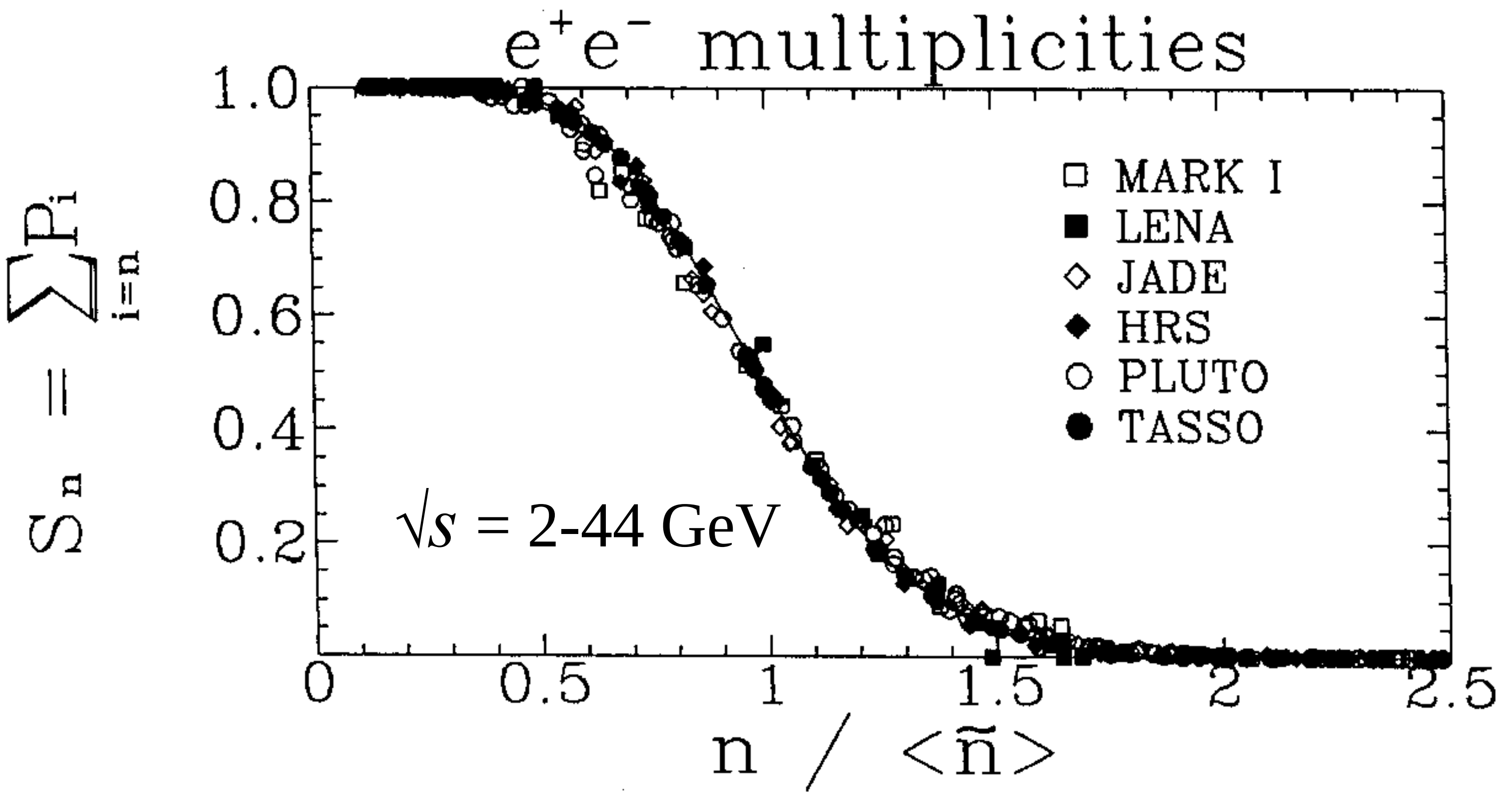
$D_{ch}{}^{pp} \sim (\langle n_{ch}{}^{pp} \rangle - 1)$



Graphical presentation of KNO-G scaling

Scaling can be graphically presented by plotting S_n [8]:

$$S_n = \sum_{k=n}^{\infty} P_k = \int_{n/\langle\tilde{n}\rangle}^{\infty} \psi(z)dz = -\phi\left(\frac{n}{\langle\tilde{n}\rangle}\right)$$



Lognormal shape of scaling function

Function fited to data on the previous plot is

$$S_n = -\phi(z)$$

with

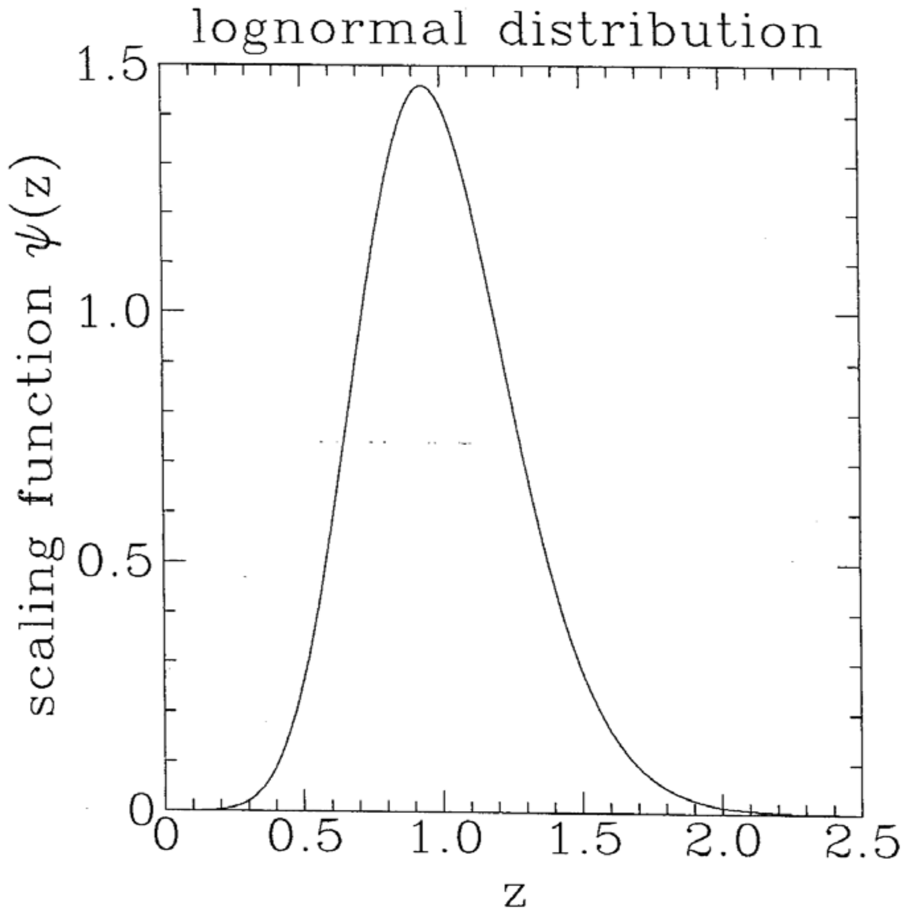
$$\phi(z) = -\int_z^\infty \psi(z)dz = -\frac{N}{2}\,\mathrm{erfc}\left(\frac{\ln(z+c)-\mu}{\sqrt{2}\,\sigma}\right)$$

and the „complementary error function”:

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

Lognormal shape of scaling function

It means that the scaling function is lognormal, i.e. that $\ln(z)$ has normal distribution [8]



$$\psi(z) = \frac{N}{\sqrt{2\pi\sigma}} \cdot \frac{1}{z+c} \exp\left(-\frac{[\ln(z+c)-\mu]^2}{2\,\sigma^2}\right)$$

Probit diagram

Using normal distribution function

$$F(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} \exp(-x^2)dx$$

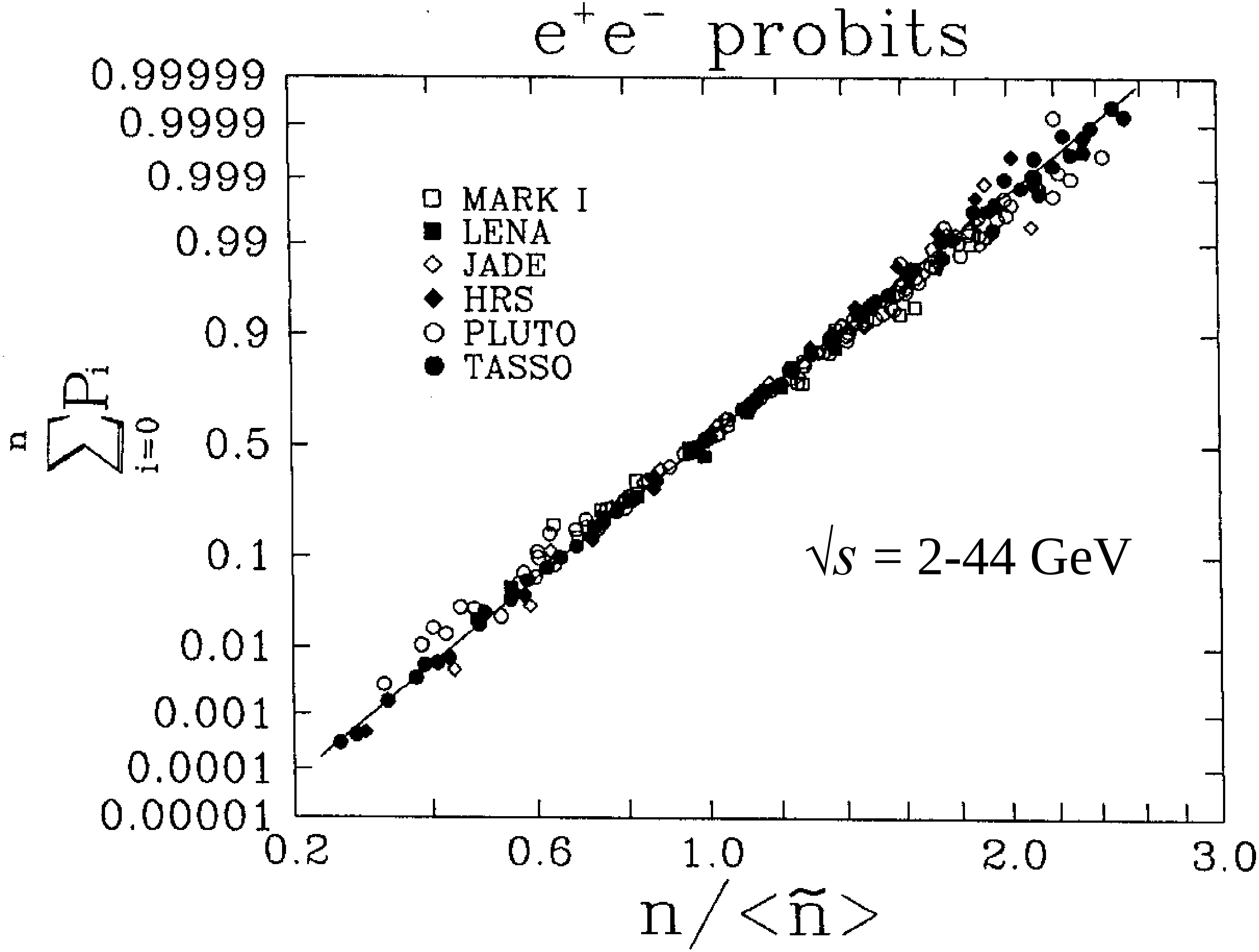
one can express the sum of P_n

$$\sum_{i=0}^{n} P_i = F\left(\frac{\ln(n+c)-\mu}{\sqrt{2}\sigma}\right)$$

Denoting $F^{-1}(x)$ as inverse function to $F(x)$, i.e. $F^{-1}(F(x))=x$ and using scaling variable $z=n/\langle n\rangle$ we get

$$F^{-1}\left(\sum_{i=0}^{n} P_i\right) = \frac{\ln(z+c')-\mu'}{\sqrt{2}\sigma}$$

Thus, $\exp(F^{-1}(\Sigma P_n))$ should be a linear function of z [9]


e+e− probits
MARK I
LENA
JADE
HRS
PLUTO
TASSO
√s = 2-44 GeV
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0.9999
0.999
0.99
0.9
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0.1
0.01
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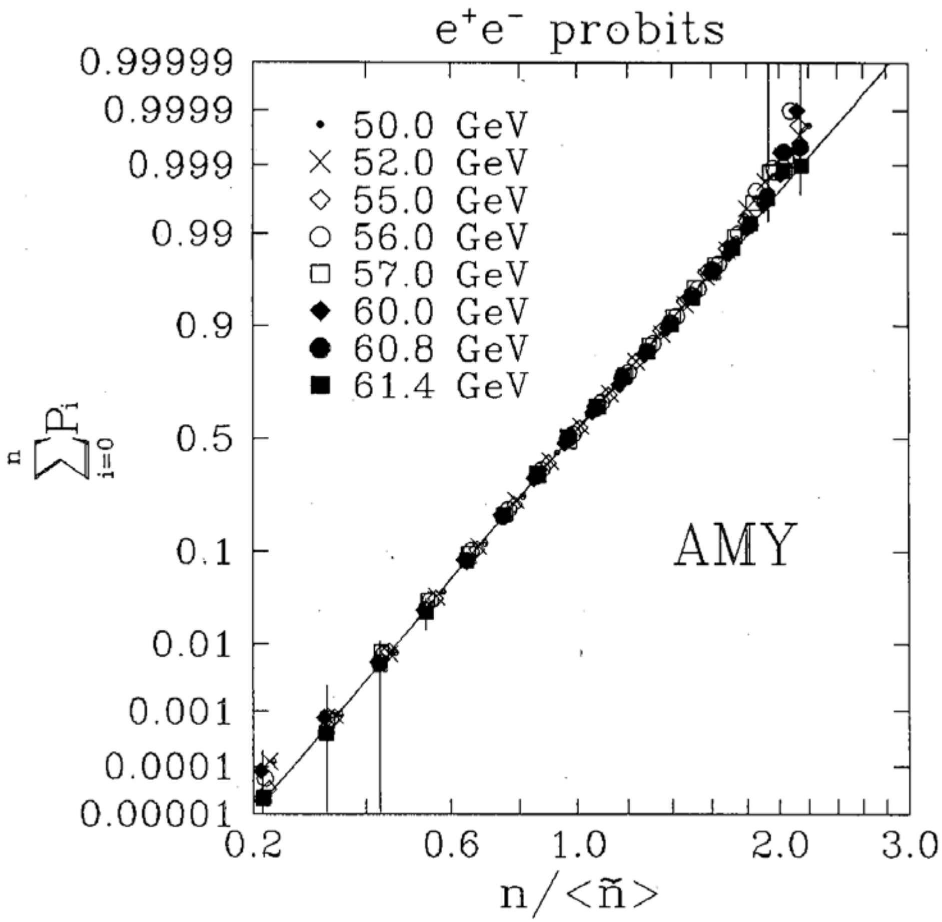
n/<ñ>
Σ_{i=0}^{n} P_i

e^+e^- probits


e+e− probits
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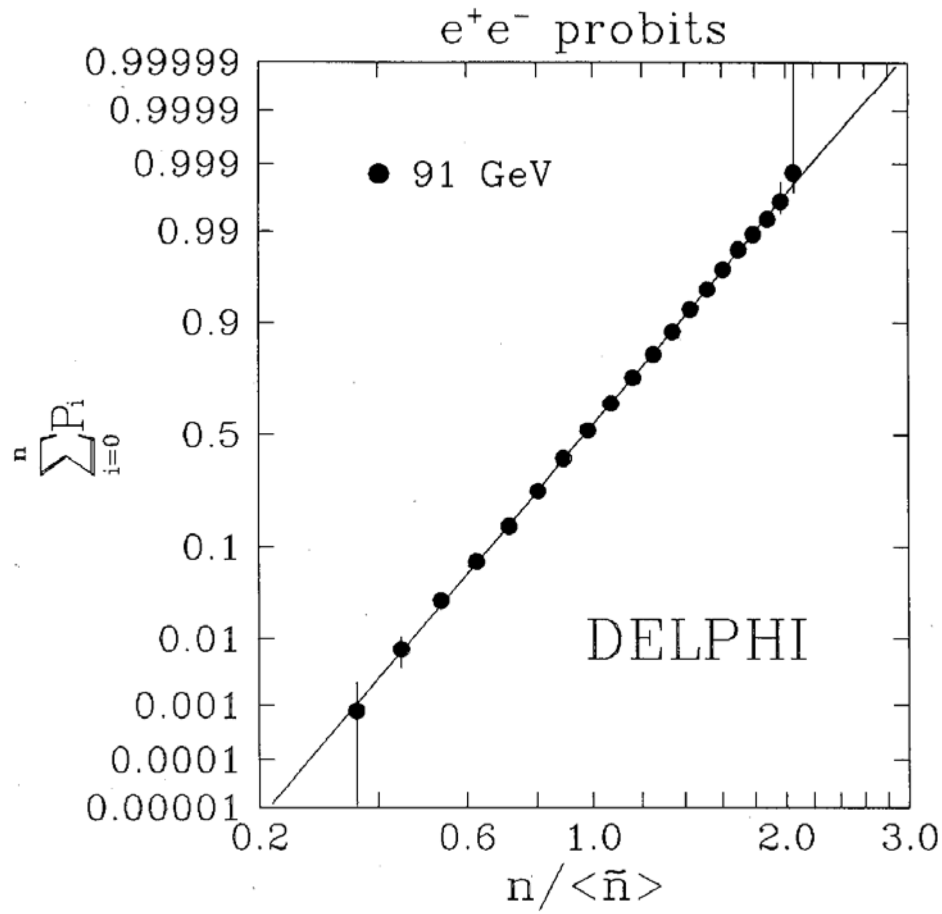
$\sum_{i=0}^{n} P_i$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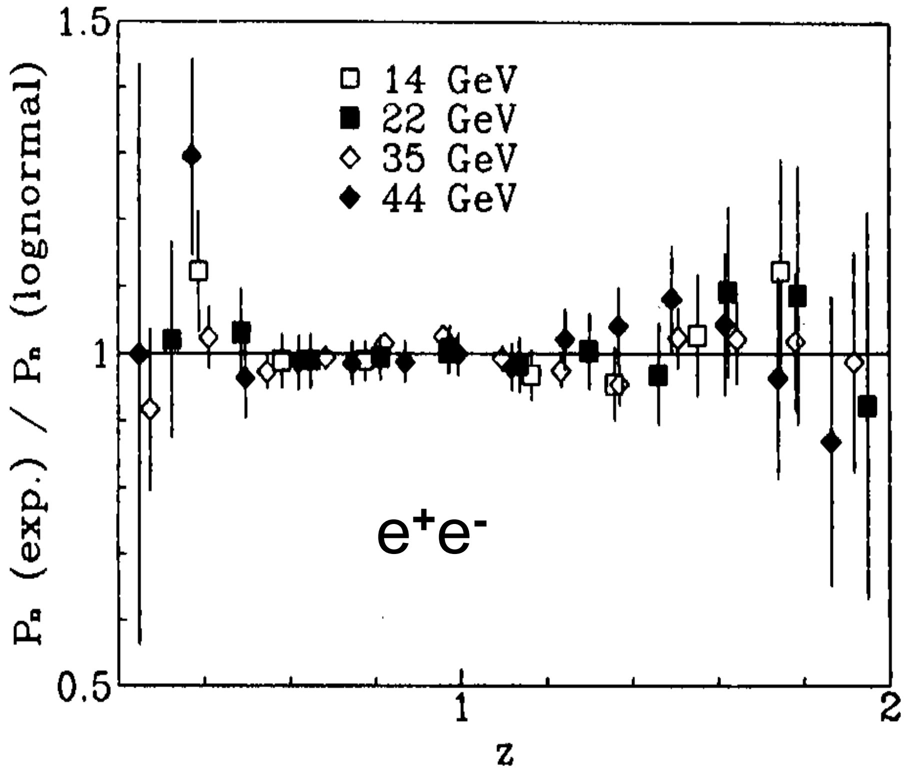
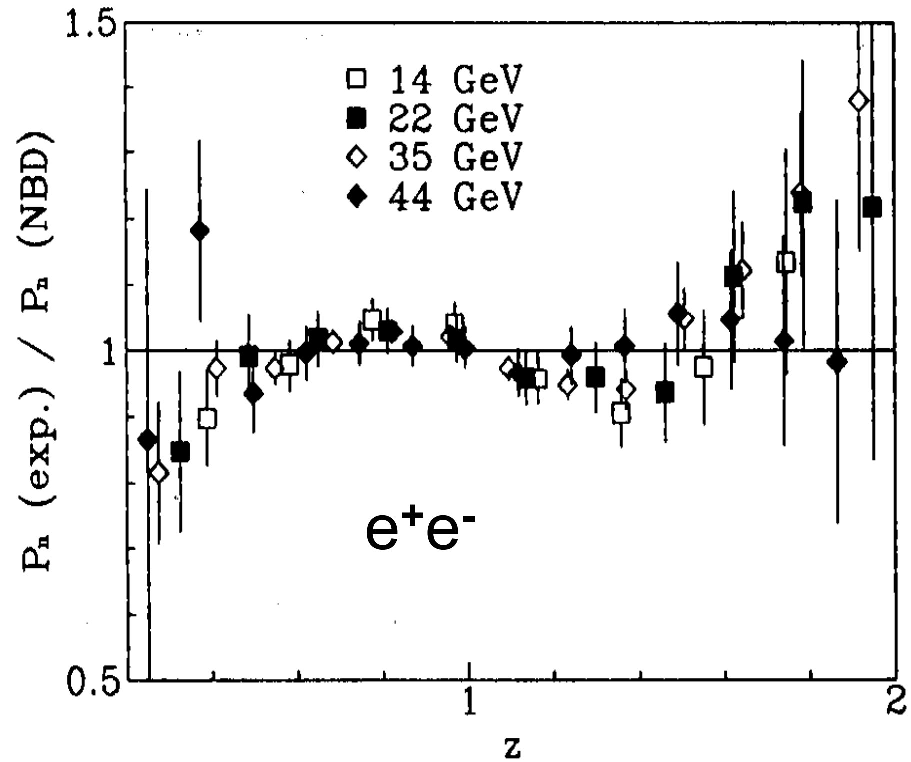
$\sum_{i=0}^{n} P_i$

lognormal or negative binomial?





Lognormal: χ^2/NDF=0.7

Common fit to all energies

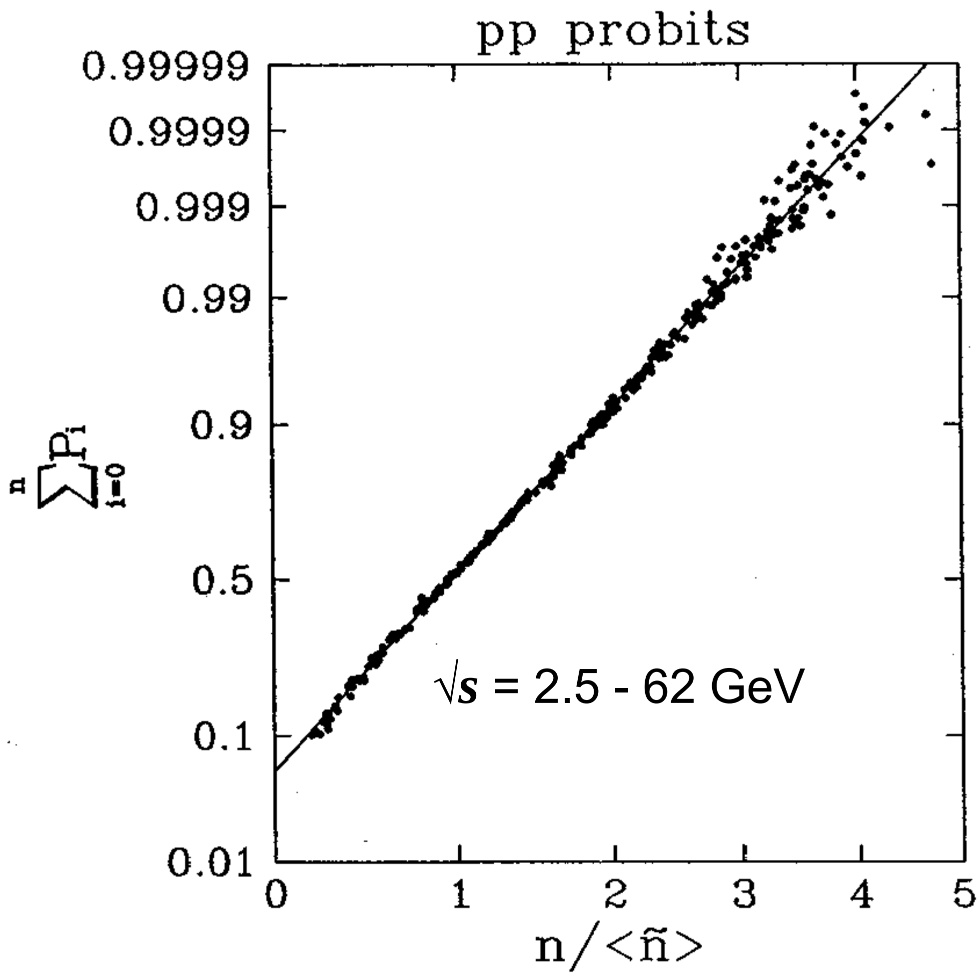
No systematic deviation

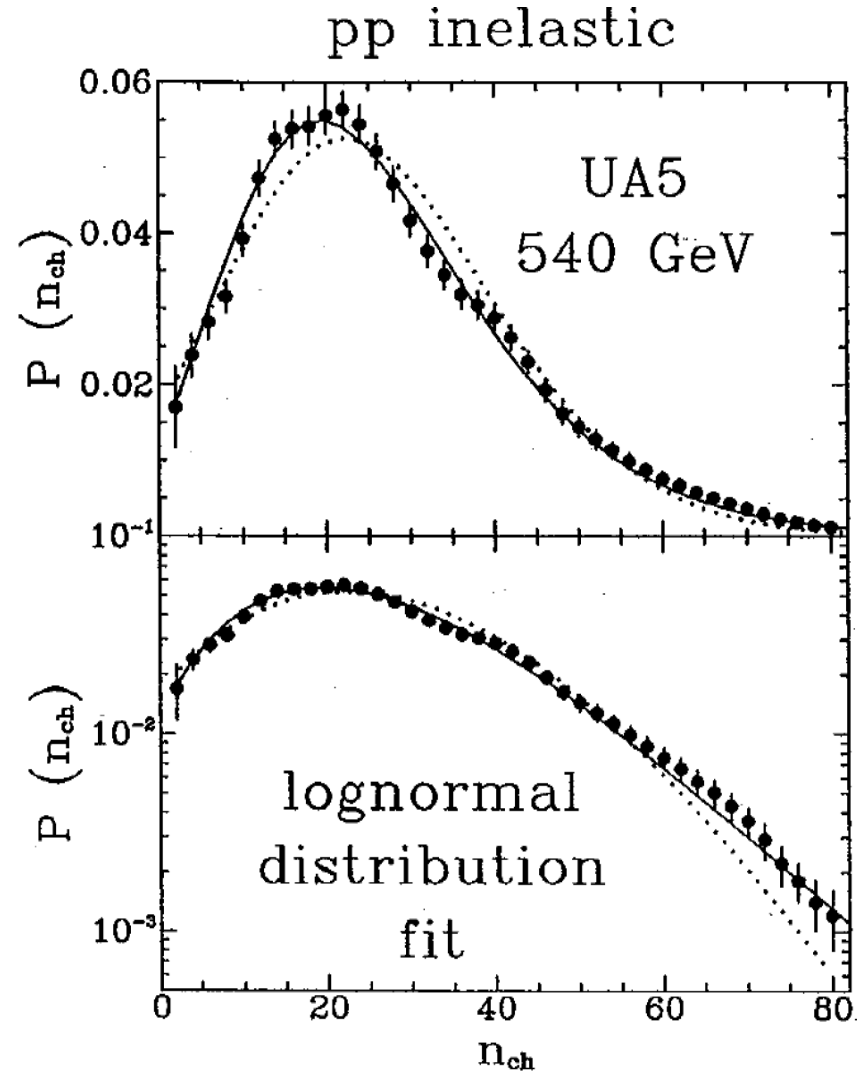
NBD: χ^2/NDF=1.5

Fit for each energy

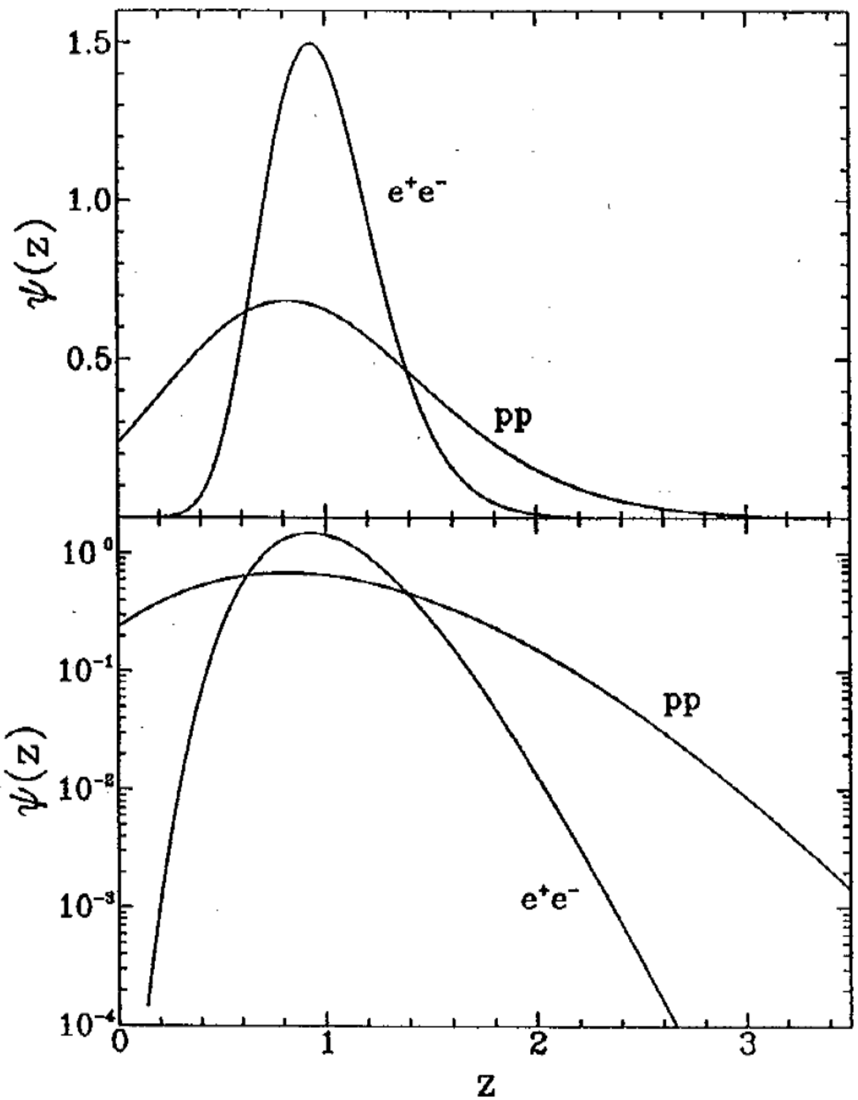
Systematic deviation seen

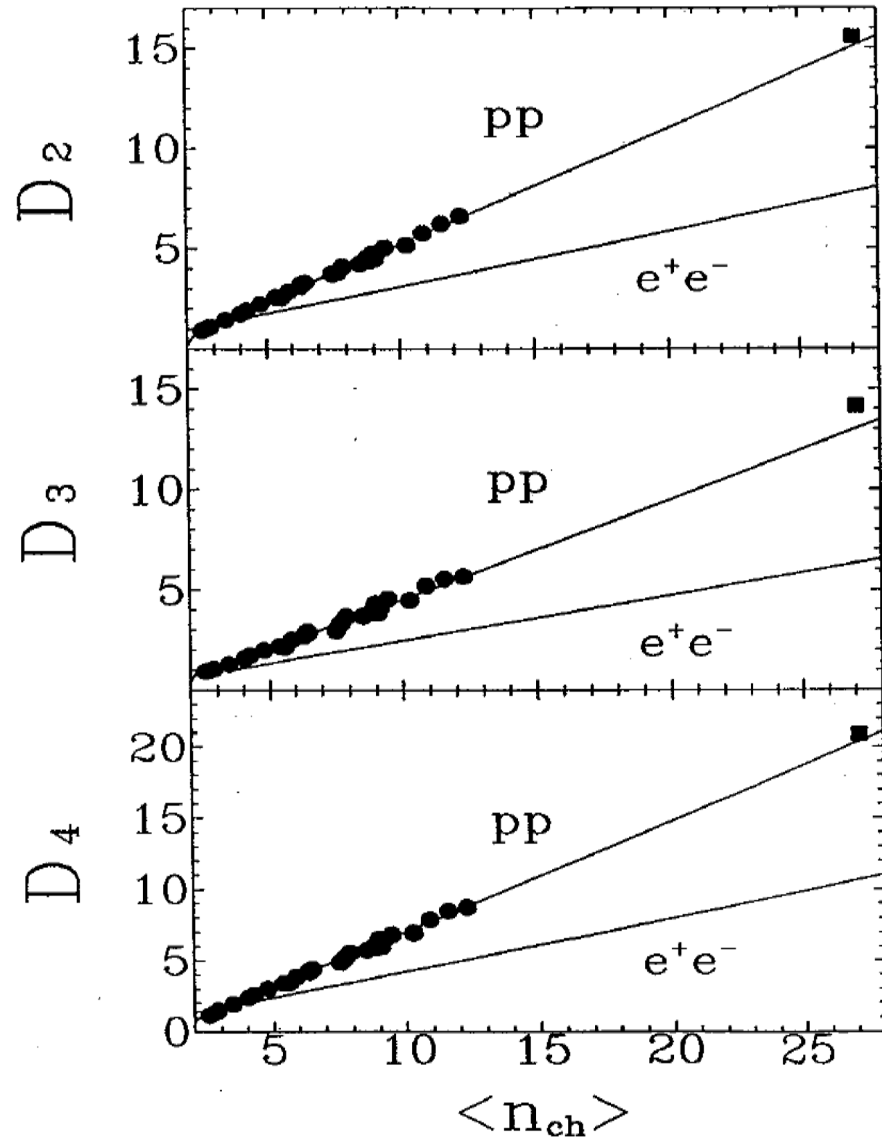
pp data scaling and its violation





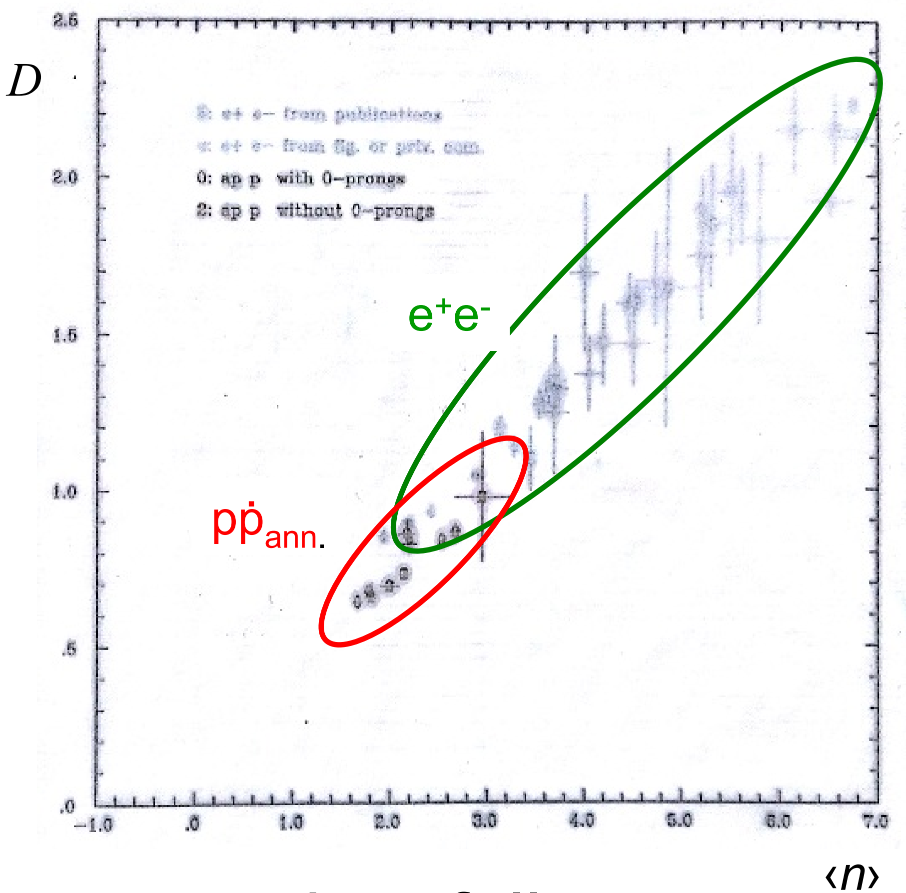
e^+e^- and pp comparision

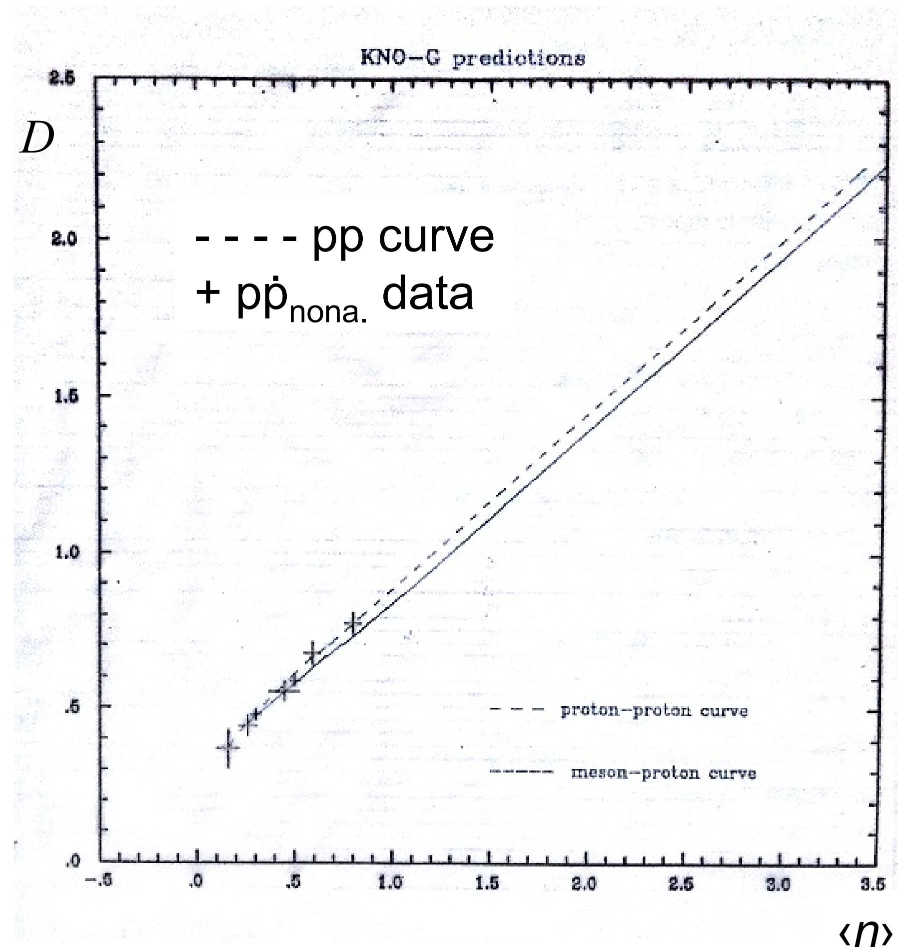




proton-antiproton data

pṗ data should be split into annihilation and nonannihilation events

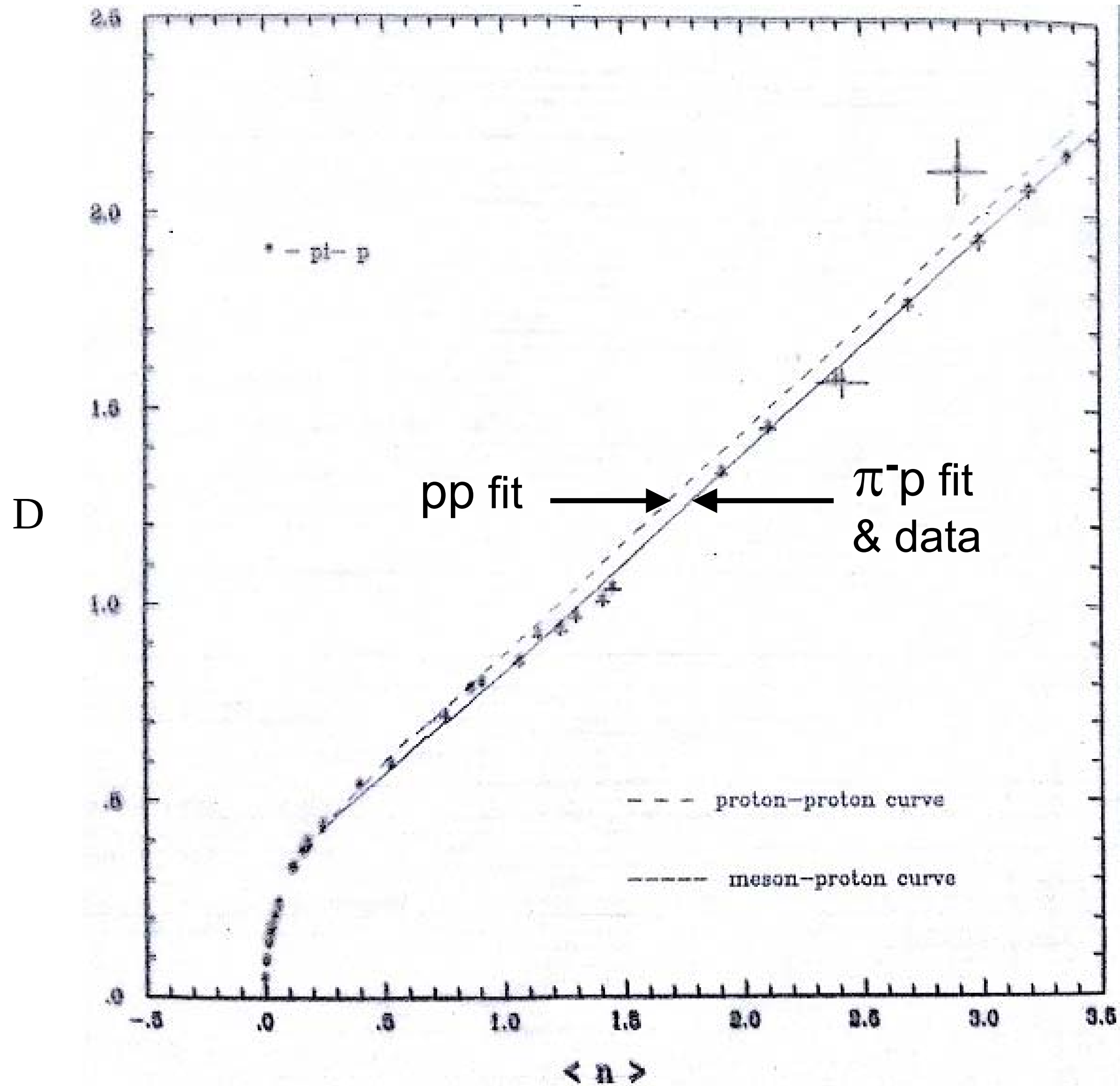




pṗ$_{ann.}$ data follow e^+e^-

pṗ$_{nona.}$ data follow pp

$\pi^{\pm}p$ and $K^{\pm}p$ data



$\pi^{\pm}p$ and $K^{\pm}p$ data

$D(\langle n \rangle)$ lines for $\pi^{\pm}$p and $K^{\pm}$p are parallel to pp

If we interpret this as a shift in n

$$D \sim (\langle n + \varepsilon \rangle + 0.5)$$

we get:

$$P_n = \int_{(n+\varepsilon)/\langle \tilde{n} \rangle}^{(n+1+\varepsilon)/\langle \tilde{n} \rangle} \psi(\tilde{n})\, d\tilde{n}$$

Indeed, using this formula one can fit all pp, pn, $K^{\pm}$p, $\pi^{\pm}$p, $\pi^{\pm}$n data with a single scaling curve

All data fitted with single lognormal curve

data	ε	χ^2/NDF	P_{lab} [GeV]
pp	0.00 ±0.01	1.26	4.0, 5.5, 6.6, 12, 19, 24, 35.7, 50, 60, 69, 100, 100, 100, 102, 175, 205, 303, 360, 400, 405, 405, 493, 1032, 1471, 2062
π^-p	-0.14 ±0.02	1.88	4.0, 5.0, 8.05, 10.0, 11.2, 13.0, 16.2, 18.5, 20, 22, 25, 50, 70, 100, 100, 147, 175, 205, 250, 360
K^-p	-0.07 ±0.03	1.54	3.0, 4.2, 8.25, 10.1, 12.6, 14.3, 16.0, 32.1, 33.8, 70, 100, 110, 147, 175
K^+p	-0.14 ±0.02	1.01	1.25, 1.32, 1.38, 1.45, 1.48, 1.96, 2.11, 2.32, 2.53, 2.72, 3.0, 3.5, 5.0, 8.25, 10, 12.7, 16, 32, 70, 100, 100, 147, 175, 200
π^-n	+0.38 ±0.05	1.60	5, 15, 21, 40, 100, 205, 360
pn	-0.37 ±0.05	1.77	3.83, 5.09, 6.1, 19.2, 28, 100, 195, 200, 300, 400

Why lognormal?

- Normal distribution appears in additive proceses, where changes Δx of random variable x are independent from the value of x

$$x_{i+1} = x_i + \Delta x_i$$

- Central Limit Theorem says that distribution of x is normal regardless the distribution of Δx
- For multiplicative processes, Δx is proportional to the value of x

$$x_{i+1} = x_i + \epsilon_i x_i$$

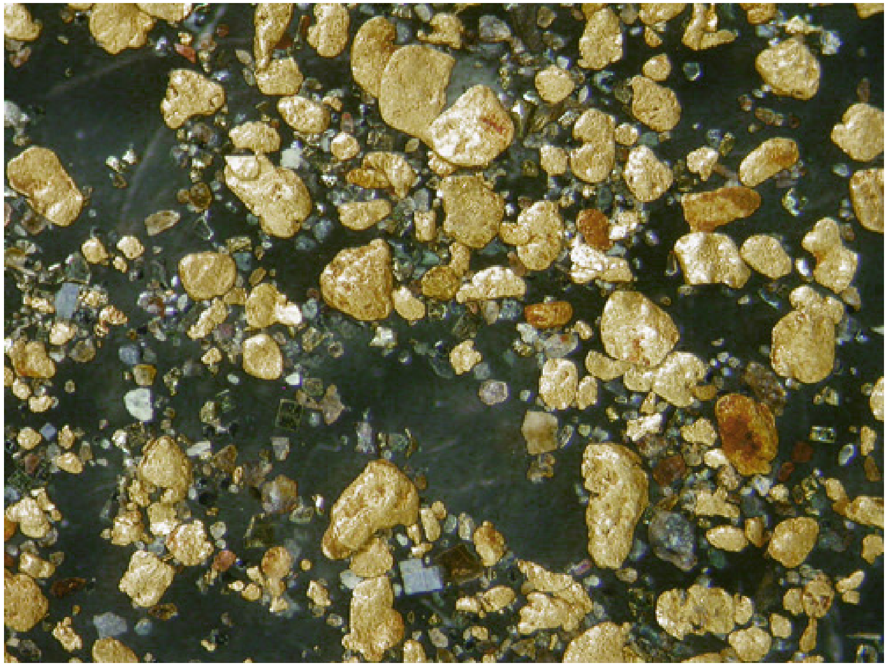
$$x = x_0 \prod_i (1 + \epsilon_i)$$

$$\ln(x/x_0) = \sum_{i=1} \epsilon_i$$

so, the $\ln(x)$ has normal distribution.

Multiplicative processes

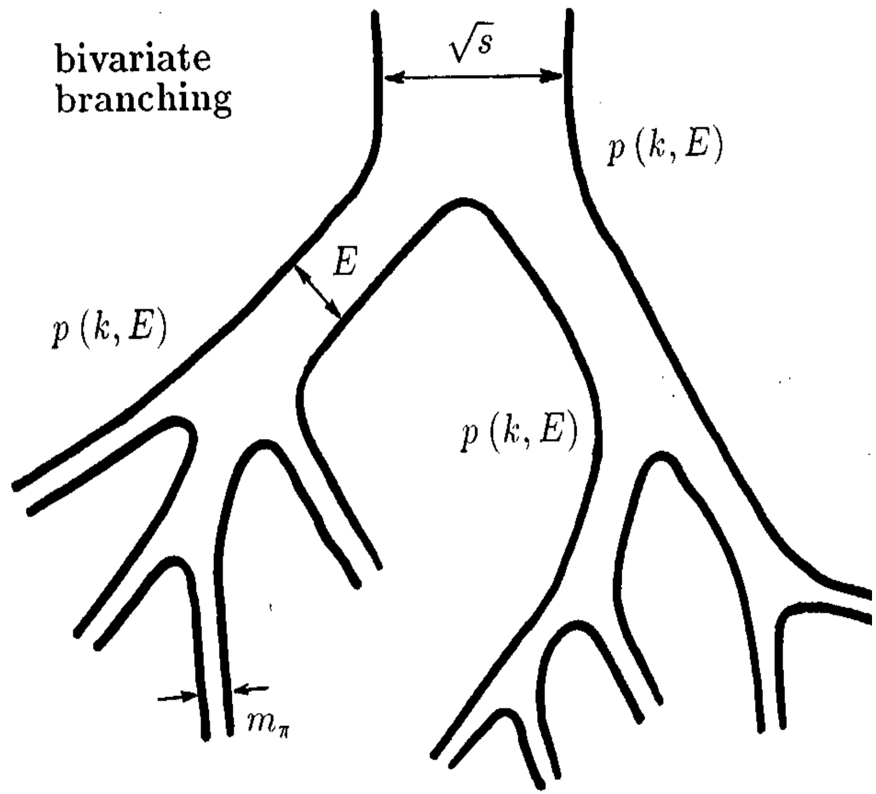
- One of the first applications of lognormal distribution was gold mining
- Weight of gold grains was found to have lognormal distribution [4]
- It was explained by Kolmogorov [5] assuming that smaller grains are created by consequtive decays of larger grains
- This an example of stochastic branching (or cascade) process



Particle production as branching process

QCD suggests that particles are produced by a kind of branching process:

- In $e^+e^- \rightarrow q\bar{q}$ reaction the $\sqrt{s}$ energy is converted into potential energy of the string between quarks
- The string brakes into 2 qq strings dividing the energy
- Each qq string has a chance to decay further if the energy is enough to form new particles



Particle production as branching process

Number of particles in i-th iteration n_i is proportional to $n_{i\text{-}1}$

$$n_i = n_{i\text{-}1} \cdot (1+ \epsilon_i)$$

The energy per particle E_i changes because

- it is divided between more particles (factor $1 + \epsilon_i$)
- Fraction δ_i of it is converted into kinetic energy

$$E_i = E_{i\text{-}1} \cdot (1\text{-}\ \delta_i) \,/\, (1+ \epsilon_i)$$

Hence, n_i and E_i will have bivariate lognormal distribution

Lognormal distribution of n_i and E_i

$$f\left(\frac{\tilde{n}}{\tilde{n}_0}, \frac{E}{E_0}\right) = \frac{1}{2\pi\sigma_1\sigma_2 \frac{\tilde{n}}{\tilde{n}_0}\frac{E}{E_0}\sqrt{1-\rho^2}} \cdot \exp$$

$$\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{\left(\ln\frac{\tilde{n}}{\tilde{n}_0}-\mu_1\right)^2}{\sigma_1^2} - 2\rho\frac{\left(\ln\frac{\tilde{n}}{\tilde{n}_0}-\mu_1\right)\left(\ln\frac{E_0}{E}-\mu_2\right)}{\sigma_1\sigma_2} + \frac{\left(\ln\frac{E_0}{E}-\mu_2\right)^2}{\sigma_2^2}\right]\right\}$$

Fixing E_0 at $\sqrt{s}$ and final energy E at the pion mass m_π we get multiplicity distribution density function as the conditional probability:

$$f(\tilde{n}|\sqrt{s}) = \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}\,\tilde{n}} \cdot$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)\,\sigma_1^2}\left[(\ln\tilde{n}-\mu_1) - \rho\frac{\sigma_1}{\sigma_2}\left(\ln\frac{\sqrt{s}}{m_\pi}-\mu_2\right)\right]^2\right\}$$

Average multiplicity vs $\sqrt{s}$

Integrating $\tilde{n} \cdot f(\tilde{n}|\sqrt{s})$ we get the dependence of average multiplicity $\langle\tilde{n}\rangle$ on collision energy $\sqrt{s}$

$$\langle\tilde{n}\rangle = \exp\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}\left(\ln\frac{\sqrt{s}}{m_\pi} - \mu_2\right) + \frac{1}{2}\sigma_1^2\left(1-\rho^2\right)\right]$$

Simplifying:

$$\langle\tilde{n}\rangle = \exp\left(\alpha\ln s + \gamma\right)$$

or even simpler:

$$\boxed{\langle\tilde{n}\rangle = \tilde{\beta}\, s^{\alpha}}$$

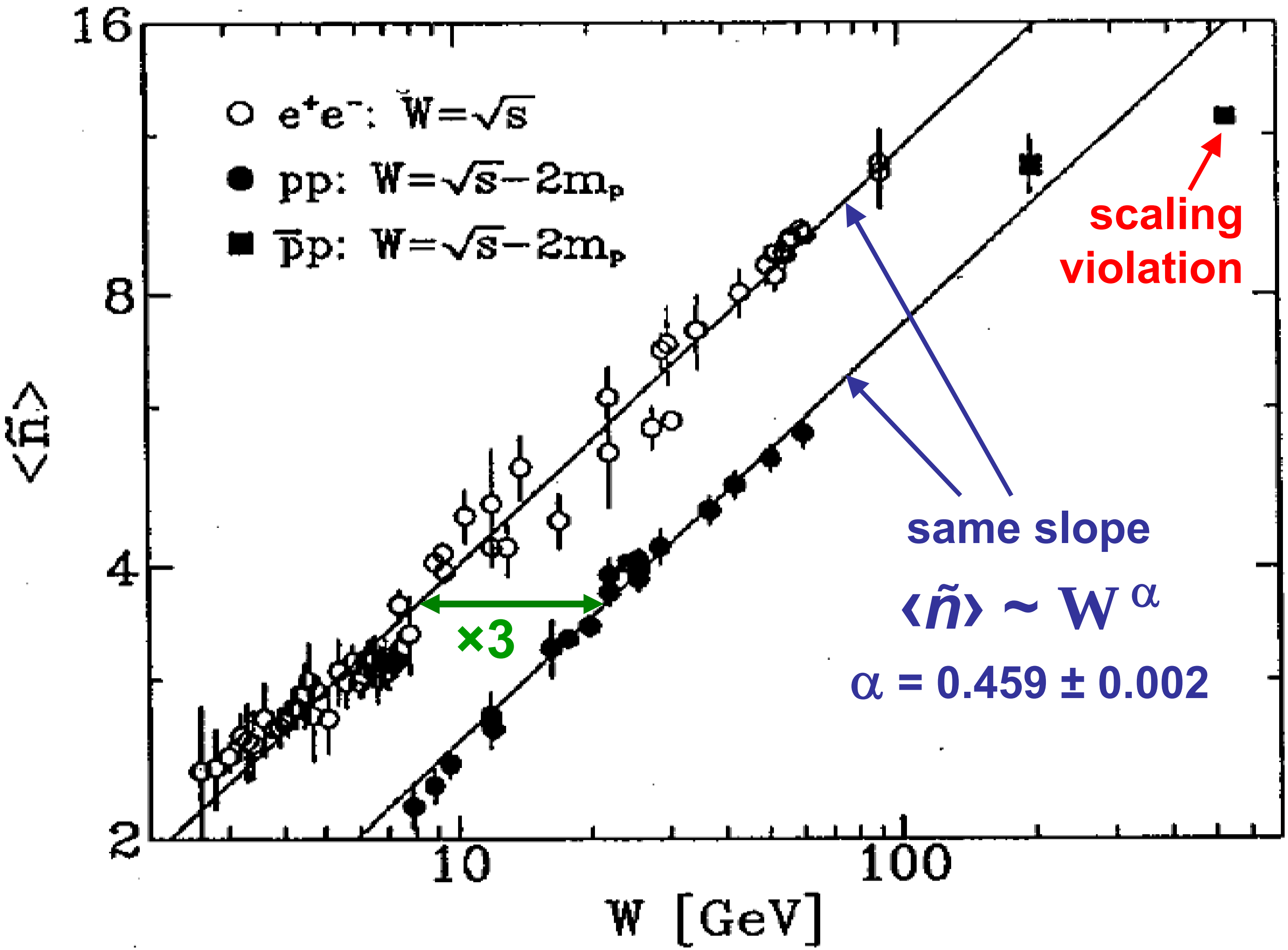
We can test it with data using approximate formula:

$$\langle n_{ch}\rangle \approx 2\left(\langle\tilde{n}\rangle - 0.5\right)$$

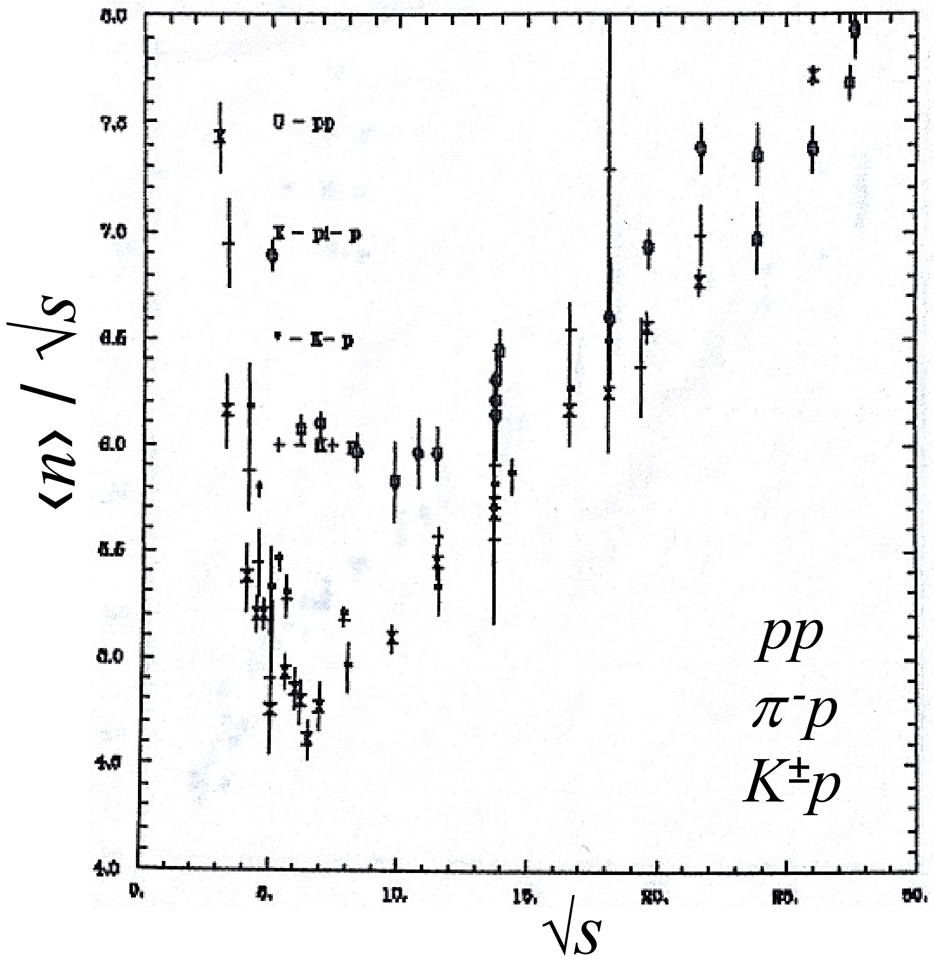
so we get:

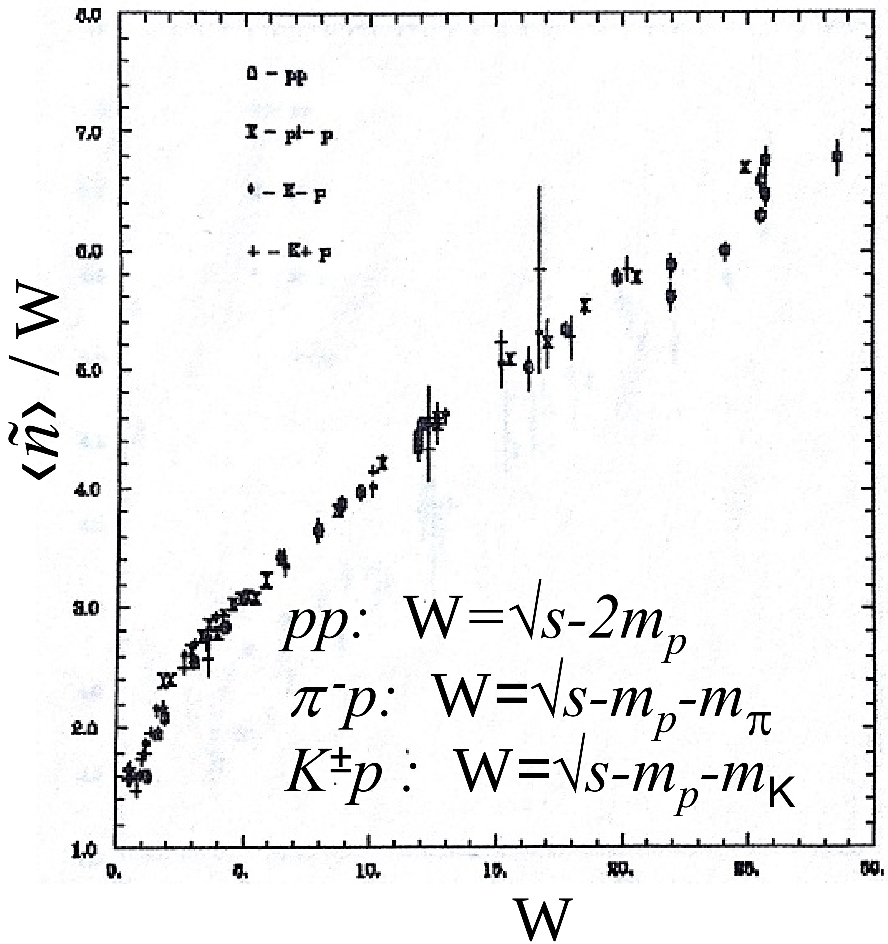
$$\langle n_{ch}\rangle = \beta\, s^{\alpha} - 1$$

Average multiplicity vs $\sqrt{s}$



‹n›(√s) for π±p and K±p





‹n›(√s) has the same shape for *pp, π⁻p* and *K±p*

if expressed in proper scaling variables $\langle \tilde{n} \rangle$(W)

Scaling for pp and its violation

pp collisions are more complicated than e^+e^- [10]

1. Only a fraction $\kappa \cdot \sqrt{s}$ carried by colliding partons is available for particle production
2. More than one, (say k) pairs of partons can collide

One can prove that the scaling holds if κ and k distributions do not depend on energy

- In such case $\langle \tilde{n} \rangle(\sqrt{s})$ for pp and e^+e^- should have the same slope

Physical reasons for scaling violation could be:

- $k=1$ for $\sqrt{s} < 500$ GeV, $k \neq$ const ($\sqrt{s} > 500$ GeV)
- for $\sqrt{s} > 500$ GeV gg collisions become important with κ_{gg} distribution different from κ_{qq}

Conclusions

Considering particle production as a branching process gives:

- lognormal shape of the universal (scaling) distribution
  - one curve for all e^+e^- and $p\bar{p}_{ann.}$ data
  - one curve for all pp, np, $K^{\pm}p$, π^-p and π^-n data ($\pm\varepsilon$)
- simple energy dependence of average multiplicity, common for e^+e^- and all hh collisions (with $\sqrt{s}/3$ for pp, and $+\varepsilon$ for other hh)
- scaling violation for pp $\sqrt{s}$ > 500 GeV explained as multiple qq or gg collisions

Reversing the argument:

- Observed properties of multiplicity distributions confirm QCD-suggested picture that particle production is a scale-invariant branching process

References

1. A.K.Wróblewski: Acta Phys. Pol. B4 (1973) 857
2. Z.Koba, H.B.Nielsen and P.Olesen: Nucl. Phys. B40 (1972) 317.
3. A.I.Golokhvastov: Sov. J. Nucl. Phys. 27 (1978) 430; 30 (1979) 128
4. Sovietskaja Zolotopromyshlennost 12 (1937) 39.
5. A.N.Kolmogorov: Comptes Rendus de l'Academie des Sciences de l'URSS (Doklady Akademii Nauk SSSR) 31 (1941) 99
6. R.Szwed and G.Wrochna: "New ISR and SPS Collider Multiplicity Data and the Golokhvastov Generalization of the KNO Scaling", Zeitschrift für Physik C29 (1985) 255.
7. R.Szwed and G.Wrochna: "Scaling Predictions for Multiplicity Distributions at LEP", Zeitschrift für Physik C47 (1990) 449.
8. R.Szwed, G.Wrochna and A.K.Wróblewski: "Genesis of the Lognormal Multiplicity Distribution in the e+e− Collisions and Other Stochastic Processes", Modern Physics Letters A5 (1990) 1851.
9. R.Szwed, G.Wrochna and A.K.Wróblewski: "New AMY and DELPHI Multiplicity Data and the Lognormal Distribution", Modern Physics Letters A6 (1991) 245.
10. M.Gaździcki, R.Szwed, G.Wrochna and A.K.Wróblewski: "Scaling of Multiplicity Distributions and Collision Dynamics in e+e− and pp Interactions", Modern Physics Letters A6 (1991) 981.

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