MYSTERY OF THE NEGATIVE BINOMIAL DISTRIBUTION*

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Negative binomial distribution has been used to describe data on multiplicities of particles in inelastic and non-diffractive pp collisions and also $\bar{p}p$ interactions as measured by the UA5 Collaboration at the CERN SPS Collider. Various incorrect procedures and misinterpretations encountered in the literature on the subject are pointed out. The negative binomial distribution is found to have serious shortcomings which cast doubts on its usefulness in describing and interpreting experimental data.

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1. *Introduction*

The Negative Binomial Distribution (NBD) has been recently promoted [1] to the status of a new empirical law describing multiplicity distributions of particles in high energy collisions. In the present paper we critically examine experimental evidence for this claim.

The NBD belongs to the family of the Poisson transforms of some important probability density functions frequently used in statistical physics [2].

In Table I we remind the most popular of them [2]. The simplest distribution function with two free parameters is the NBD. Because of its flexibility it is well suited to describe various pieces of data on particle multiplicities in high energy collisions [1, 3–5]. However it must be stressed that in many cases we observe misunderstandings both in the way of applying the NBD to describe multiplicity distributions and in the interpretation of results [6].

The aim of the present paper is threefold. First, to set in order the ways of applying the NBD for multiplicity distributions. Secondly, to clarify a few common misunderstandings about interpretation of NBD-fits to the data. Thirdly, to answer the question whether the NBD can really be treated as a new empirical law for high energy collisions. The discussion is based on the pp multiplicity data and SPS Collider $\bar{p}p$ data.

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TABLE I

Poisson transforms $P(n) = \int_0^\infty \frac{x^n e^{-x}}{n!} f(x)dx$

$f(x)$	$P(n)$	
$\delta(x-\bar{n})$	$\frac{\bar{n}^n e^{-\bar{n}}}{n!}$	Poisson
$\frac{1}{\bar{n}} e^{-x/\bar{n}}$	$\frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$	Geometric (Bose-Einstein)
$\frac{1}{\Gamma(k)} (\bar{n}/k)^{-k} x^{k-1} e^{-kx/\bar{n}}$	$\begin{bmatrix} n+k-1 \\ n \end{bmatrix} \frac{(\bar{n}/k)^n}{(1+\bar{n}/k)^{n+k}}$	Negative Binomial
$\frac{1}{\bar{n}-\mu} \exp\left[-\frac{x+\mu}{\bar{n}-\mu}\right] I_0\left[\frac{2\sqrt{\mu x}}{\bar{n}-\mu}\right]$	$\frac{(\bar{n}-\mu)^n}{(1+\bar{n}-\mu)^{n+1}} \exp\left[\frac{-\mu}{1+\bar{n}-\mu}\right] L_n\left[\frac{-\mu}{(1+\bar{n}-\mu)(\bar{n}-\mu)}\right]$	Laguerre
$\frac{k}{\bar{n}-\mu}\left[\frac{x}{\mu}\right]^{\frac{k-1}{2}} \exp\left[-k\frac{x+\mu}{\bar{n}-\mu}\right] \times I_{k-1}\left[\frac{2k\sqrt{\mu x}}{\bar{n}-\mu}\right]$	$\frac{[(\bar{n}-\mu)/k]^n}{[1+(\bar{n}-\mu)/k]^{n+1}} \exp\left[\frac{-\mu}{1+(\bar{n}-\mu)/k}\right] \times L_n^{k-1}\left[\frac{-\mu k/(\bar{n}-\mu)}{1+(\bar{n}-\mu)/k}\right]$	Laguerre-order k

Experimentally, multiplicity distribution in full phase-space is measured for fully inclusive sample corresponding to the total inelastic cross-section called hereafter "inelastic sample" [7, 8] and for "non-diffractive sample" corresponding to the full inclusive sample with single diffractive events excluded [9–11].

It should be stressed that the non-diffractive data samples usually have large uncertainties. Strictly speaking, it is impossible to subtract experimentally single diffractive events from the inelastic sample in an unbiased way. Hence the non-diffractive samples are rather scarce and moreover have considerable systematic errors, especially for the lowest multiplicities. Thus, any tests based on the non-diffractive data are of poorer quality than those based on data from the inelastic sample. In some cases, discussed in this paper, the tests of theoretical hypotheses are positively passed by the non-diffractive data, contrary to the inelastic sample, only because the former are much less accurate and therefore do not impose severe enough constraints. Unfortunately, this fact is often ignored.

2. *Genuine multiplicity measure*

Customarily, the number of charged particles n_{ch} is used as a multiplicity measure, since it can be easily determined in track detectors.

However, it is well known that the number of prongs is not a good multiplicity measure since it includes initial charges of colliding particles and moreover, due to charge conservation in each interaction, positive and negative partners are totally correlated. Depending on the initial charges the number of charged particles is always even ($n_{ch} = 0, 2, 4, 6 \ldots$) or always odd ($n_{ch} = 1, 3, 5, 7 \ldots$) and therefore does not meet the requirement of probability distribution functions which are defined for all non-negative integers ($n = 0, 1, 2, 3 \ldots$).

Instead, the genuine multiplicity measure should be used, denoted hereafter by n ($n = 0, 1, 2, 3 \ldots$). It can be defined as the number of acts of pair creation in the event, where by a pair we denote a positive and a negative particle correlated by charge conservation.

The genuine multiplicity measure can be always converted into the n_{ch} for a given type of collision. For the pp collisions it is easy to see that the number of negative particles n_- in the final state coincides with genuine multiplicity n:

$$n_- = n \quad \text{and} \quad n_{ch} = 2n+2.$$

Hence

$$\langle n_{ch} \rangle = 2\langle n \rangle + 2.$$

We should remember that at the highest energies (SPS Collider) the $\bar{p}p$ data are used instead of the pp. This is allowed because the $\bar{p}p$ multiplicity data are similar to the pp ones providing the annihilation is subtracted [12]. At high enough energies annihilation contribution can be neglected. Therefore multiplicities for $\bar{p}p$ collisions can be counted in the same way as for the pp data ($n_{ch} = 2, 4, 6 \ldots$).

Evidently, one can use probability distributions of charged particles $p(n_{ch})$ or $P(n)$ to describe multiplicity distributions. There is exactly the same information content in both of them. However, when formulas such as the Poissonian or the NBD are used to describe multiplicity distributions only the genuine multiplicity can be used since these formulas give probabilities for all non-negative integers n. Unfortunately, very often $P(n_{ch})$ is used which leads to much confusion when results of such an incorrect procedure are furthermore interpreted. (See Section 4 and 6).

3. Parameters of the NBD and full phase-space multiplicity data

Let us suppose that a given multiplicity distribution is characterized by genuine multiplicity n ($n = 0, 1, 2, 3 \ldots$) and we attempt to describe this distribution by the NBD. The NBD:

$$P(n) = \binom{n+k-1}{n} \frac{(\bar{n}/k)^n}{(1+\bar{n}/k)^{n+k}} \tag{1}$$

has two free parameters $\bar{n}$ and k varying with energy. Parameter $\bar{n}$ has an interpretation of the average multiplicity. The dispersion $D_{NBD} = (n^2 - \bar{n}^2)^{1/2}$ obeys the following relation:

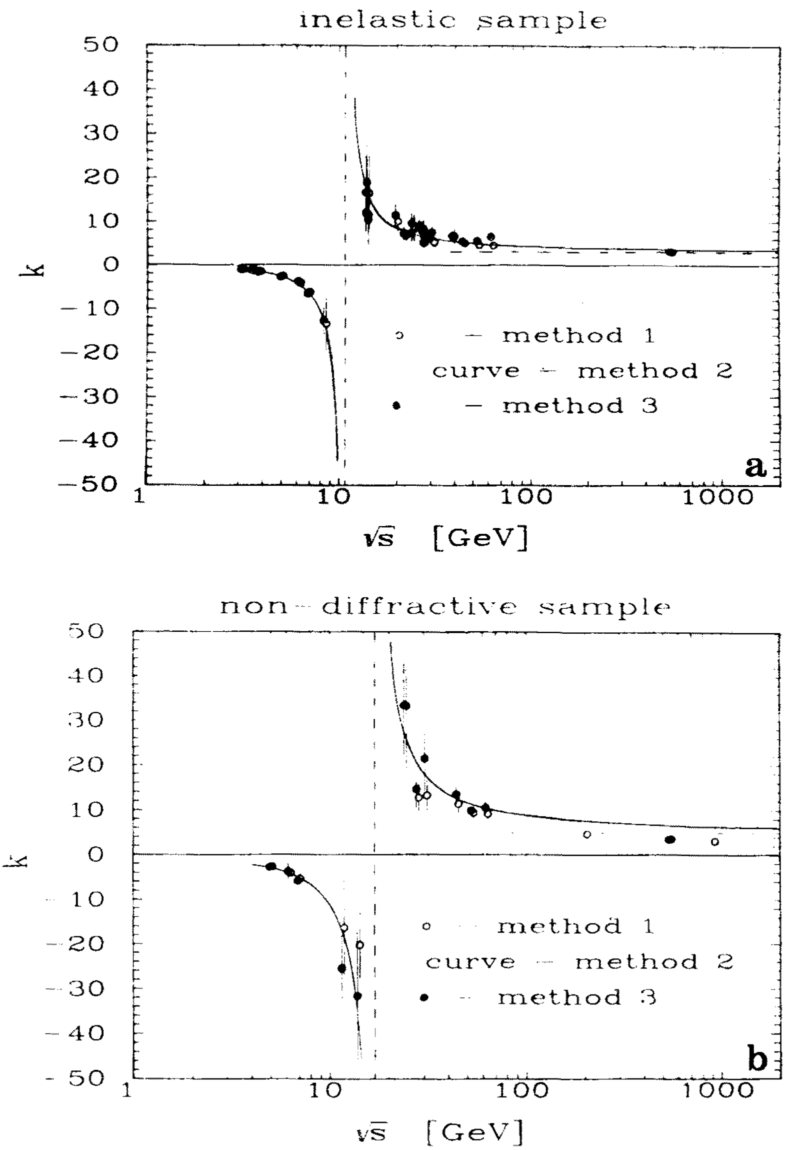
$$D^2_{NBD} = \bar{n} + \bar{n}^2/k. \tag{2}$$

To distinguish between the theoretical values of average multiplicity $\bar{n}$ and dispersion D_{NBD} and the experimental ones, we denote the latter by $\langle n \rangle$ and D.

We can think of at least four methods of choosing the best set of parameters $\bar{n}$ and k to describe a given multiplicity distribution.

Method 1. Assuming that $\bar{n}$ and D_{NBD} for the NBD are equal to the experimentally measured $\langle n \rangle$ and D, we get:

$$\bar{n} = \langle n \rangle, \quad k = \frac{\langle n \rangle^2}{D^2 - \langle n \rangle}. \tag{3}$$



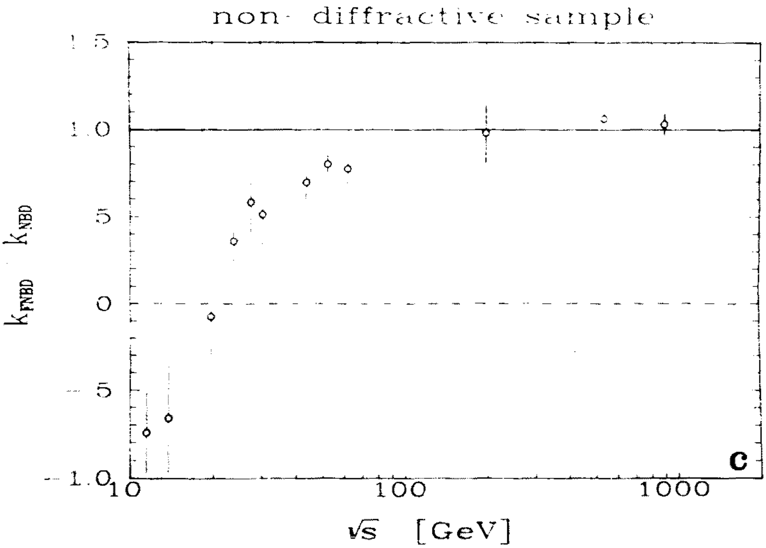


Fig. 1 a, b. The k parameter of the NBD fits obtained by Method 1 (open circles), Method 2 (solid curve) and Method 3 (full circles) for inelastic (a) and non-diffractive (b) event samples. c — The ratio of k_{NBD}/k_{FNBD}

Method 2. We can use the well-known empirical formula connecting D and $\langle n \rangle$ [13]:

$$D = A(\langle n \rangle + B). \tag{4}$$

If by n we denote the number of negative particles then for pp data $B \approx 1/2$. If the NBD holds ($\bar{n} \approx \langle n \rangle$) this leads to the analytical relation between k and $\langle n \rangle$:

$$1/k \cong A^2 + \frac{2A^2B-1}{\langle n \rangle} + \frac{A^2B^2}{\langle n \rangle^2}. \tag{5}$$

Method 3. We can directly fit the NBD to the data and obtain $\bar{n}$ and k (in this case the fitted value of $\bar{n}$ does not necessarily equal $\langle n \rangle$).

Method 4. We can assume that the parameter $\bar{n}$ is equal to the experimental average multiplicity ($\bar{n} = \langle n \rangle$), and fit the NBD to the data with only one free parameter k.

Under the assumption that the NBD describes well multiplicity distributions, the four methods should give similar results.

Figure 1a illustrates the results of the Methods 1, 2 and 3 applied to the inelastic multiplicity data [7, 8], and Fig. 1b the corresponding results for the non-diffractive sample (including recent UA5 data) [9–11].

The three methods give indeed consistent results. For both the inelastic and the non--diffractive sample there is a certain energy at which k grows to infinity (this corresponds to the Poisson distribution ($1/k = 0$). Below this energy $1/k < 0$ and the distributions are narrower than Poissonian.

The numerical results obtained with the Method 1 and 3 are given in Tables II and

TABLE II

NBD and inelastic sample

P_{lab}	Experiment (Method 1)			NBD fit (Method 3)		
$\sqrt{s}$	$\bar{n}$ ±	k ±	χ^2 NDF	$\bar{n}$ ±	k ±	χ^2 NDF
4.0	0.272	−1.09	0	0.271	−1.07	0
3.1	0.013	0.37	3	0.000	0.00	1
5.5	0.352	−1.31	11	0.349	−1.23	0
3.5	0.005	0.13	3	0.000	0.00	1
6.6	0.429	−1.49	45	0.404	−1.62	24
3.8	0.015	0.35	5	0.038	0.31	3
12.0	0.716	−2.58	171	0.706	−2.79	122
4.9	0.006	0.19	5	0.033	0.41	3
19.0	1.008	−4.21	46	0.991	−3.75	42
6.1	0.012	0.43	7	0.038	0.72	5
24.0	1.123	−6.17	34	1.114	−6.58	30
6.8	0.010	0.86	8	0.018	0.67	6
35.7	1.391	−13.5	6	1.387	−12.8	6
8.3	0.020	5.8	7	0.026	3.0	5
50.0	1.679	−71.0	7	1.696	−31.0	7
9.8	0.055	353.0	8	0.057	31.0	6
60.0	1.795	−19.0	2	1.791	−17.7	2
10.7	0.049	18.0	8	0.021	5.3	6
69.0	1.924	39.0	23	1.949	363.0	18
11.5	0.040	70.0	10	0.038	2060.0	8
100.0	2.244	11.6	12	2.278	12.0	11
13.8	0.056	6.9	10	0.057	4.2	8
100.0	2.184	10.4	70	2.185	16.7	52
13.8	0.031	3.0	10	0.079	8.2	8
102.0	2.158	16.5	15	2.170	18.9	15
13.9	0.035	8.4	10	0.048	8.6	8
205.0	2.838	10.0	24	2.834	11.5	22
19.7	0.037	2.2	13	0.057	2.4	11
250.0	2.940	6.8	44	2.973	7.2	44
21.7	0.043	1.2	14	0.088	1.5	12
303.0	3.250	8.5	35	3.283	9.7	34
23.9	0.060	2.5	14	0.079	2.1	12
303.0	3.427	8.6	19	3.41	7.4	18
23.9	0.080	3.0	13	0.10	1.9	11
360.0	3.530	8.2	32	3.576	8.9	29
26.0	0.047	1.6	13	0.069	1.5	11

TABLE II (continued)

p_{lab} $\sqrt{s}$	Experiment (Method 1) $\bar{n}$ ±	k ±	χ^2 NDF	NBD fit (Method 3) $\bar{n}$ ±	k ±	χ^2 NDF
400.0	3.570	5.94	115	3.49	5.16	103
27.4	0.034	0.47	15	0.13	0.94	13
400.0	3.383	6.6	66	3.42	8.3	56
27.4	0.061	1.5	14	0.10	1.9	12
405.0	3.485	5.68	57	3.39	5.02	46
27.6	0.047	0.64	16	0.10	0.75	14
405.0	3.480	6.1	44	3.52	6.9	41
27.6	0.055	1.1	16	0.09	1.3	14
493.0	3.715	5.2	57	3.74	7.5	27
30.4	0.089	1.1	17	0.12	1.2	15
800.0	4.124	6.6	68	4.161	6.68	66
38.8	0.077	1.6	16	0.088	0.67	14
1032.0	4.429	5.07	17	4.486	5.48	16
44.0	0.080	0.76	19	0.057	0.55	17
1471.1	4.775	4.63	43	4.89	5.61	35
52.6	0.085	0.62	21	0.10	0.67	19
2062.0	5.13	4.51	117	5.25	6.58	67
62.2	0.11	0.65	20	0.15	0.92	18
155400.0	12.73	3.03	64	12.91	3.19	57
540.0	0.11	0.13	60	0.12	0.10	58

III. Available multiplicity data from the UA5 experiment are also included [8, 10, 11]. As seen, the NBD can be fitted with reasonable χ^2 to the inelastic sample for $\sqrt{s} > 7$ GeV. This is not the case for the data at lower energy. It is impossible to check if the non-diffractive data at low energy could be well fitted with the NBD, since as seen in Table III, only three experimental sets of data are available for $\sqrt{s} < 7$ GeV and those have probably the largest experimental biases caused by difficulties in subtracting of the diffractive component from the data. In the last column of Table III, the k-values quoted in the UA5 Collaboration papers [10, 11] are presented. For the energy $\sqrt{s} < 7$ GeV and $\sqrt{s} = 13.9$ GeV the fits have not been done in the original publication [10], therefore the presented values are obtained in this paper using the same method as the UA5 Collaboration. As seen, the k-values from the last column differ from values evaluated in this paper (middle column), especially for low energies. The reason for these differences is explained in the following Section.

TABLE III

NBD and non-diffractive sample

P_{lab} / $\sqrt{s}$	Experiment (Method 1)			NBD fit (Method 3)			FNBD fit		
	$\bar{n}$ ±	k ±	χ^2 NDF	$\bar{n}$ ±	k ±	χ^2 NDF	$\bar{n}$ ±	k ±	χ^2 NDF
12.0	0.789	−2.67	37	0.784	−2.78	27	3.435	−14.70	3
4.9	0.014	0.39	5	0.026	0.14	3	0.021	0.61	3
19.0	1.103	−4.08	129	1.073	−3.71	123	4.07	−27.0	104
6.1	0.018	0.61	7	0.094	1.71	5	0.20	26.0	5
24.0	1.255	−5.38	19	1.240	−5.89	13	4.416	−80.0	8
6.8	0.019	1.05	8	0.018	0.39	6	0.026	23.0	6
69.0	2.181	−16.0	17	2.149	−25.5	10	6.23	19.0	10
11.5	0.050	11.0	10	0.038	7.4	8	0.04	2.0	6
100.0	2.606	−20.3	28	2.560	−32.0	21	7.11	21.0	16
13.8	0.031	7.3	10	0.052	14.0	8	0.06	2.0	7
102.0	2.440	−45.0	2	2.438	−44.0	1	6.848	18.3	2
13.9	0.047	66.0	10	0.021	14.0	8	0.057	1.4	8
205.0	3.281	−112.0	32	3.282	−195.0	32	8.48	15.0	9
19.7	0.033	151.0	13	0.075	642.0	11	0.09	1.0	9
303.0	3.625	33.0	11	3.604	33.0	10	9.2	12.0	7
23.9	0.037	15.0	14	0.021	11.0	12	0.1	1.0	10
405.0	3.920	12.8	25	3.855	15.0	22	9.6	8.5	14
27.6	0.053	2.9	16	0.090	4.0	14	0.1	0.8	10
493.0	4.267	13.3	45	4.36	21.5	29	10.7	11.0	24
30.4	0.070	3.2	17	0.11	7.2	15	0.1	0.7	13
1032.0	5.039	11.4	20	5.085	13.5	16	12.2	9.4	13
44.0	0.065	2.0	19	0.065	2.1	17	0.1	0.5	17
1471.0	5.383	9.4	6	5.388	9.89	5	12.8	7.9	5
52.6	0.069	1.3	21	0.018	0.55	19	0.1	0.3	17
2062.0	5.817	9.2	29	5.818	10.6	24	13.6	8.2	24
62.2	0.082	1.2	20	0.094	1.2	18	0.1	0.4	17
21314.0	9.7	4.7					21.6	4.6	22
200.0	0.4	0.7					0.5	0.4	33
155400.0	13.18	3.63	49	13.21	3.48	45	28.3	3.69	68
540.0	0.13	0.11	49	0.12	0.10	47	0.2	0.09	66
431630.0	16.3	3.10					35.1	3.2	52
900.0	0.6	0.11					0.6	0.2	58

4. *Fake Negative Binomial Distribution*

In the recent papers [10, 11] which revived interest in the NBD the authors have tried to fit the NBD to the charged multiplicity $P(n_{ch})$. Since the NBD gives probabilities to produce $n = 0, 1, 2, \ldots$ particles, whereas $n_{ch} = 2, 4, 6, \ldots$, the authors took only even integers from the NBD ($P^{NBD}(n)$, $n \geq 2$) and then renormalized the whole distribution.

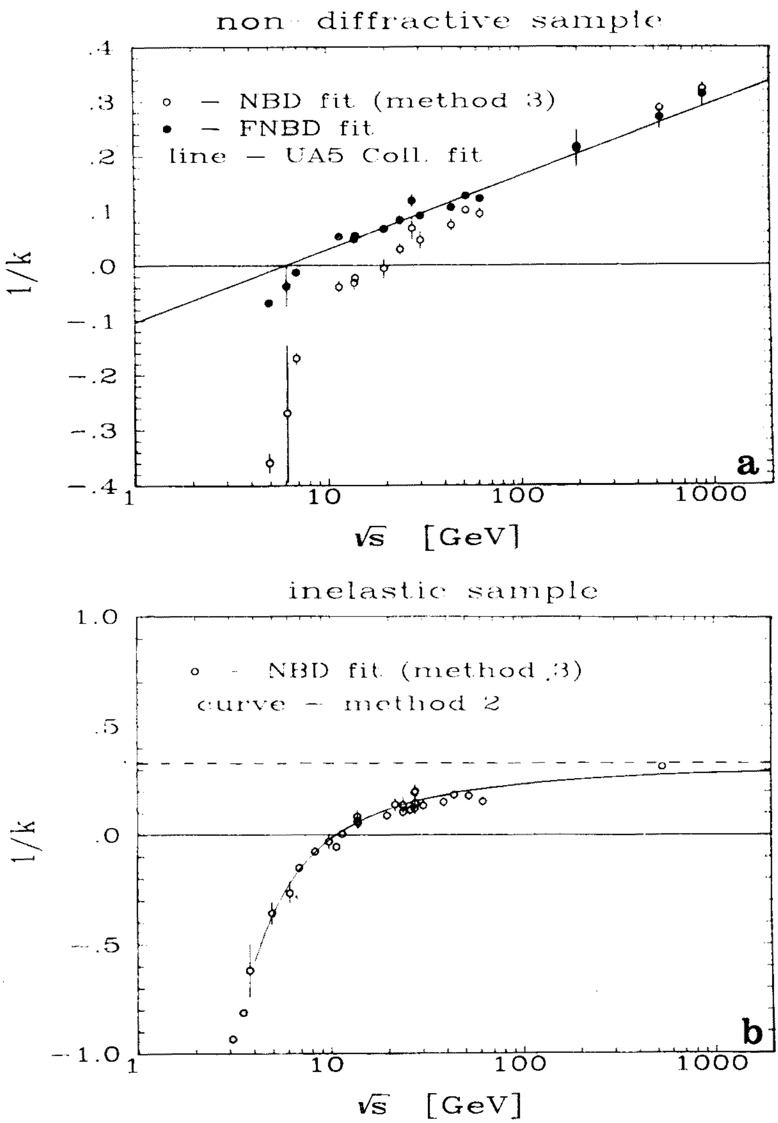


Fig. 2. a — The $1/k$ parameter of the NBD (open circles) and FNBD (full circles) fits for the non-diffractive sample. The straight line illustrates fit to the data with $\sqrt{s} < 10$ GeV, done by the UA5 Collaboration [11]. b — The $1/k$ parameter of the NBD fit for the inelastic sample. The curve corresponds to the relation (5) in the text

This procedure, strictly speaking, leads to the fitting of the different theoretical distribution, which we call hereafter Fake Negative Binomial Distribution (FNBD).

Experimentally the probability to produce $n_{ch} = 2$ in pp collisions is equal to the probability to produce $n_- = 0$ particles in the final state and should correspond to the theoretical probability for $n = 0$:

$$P^{EXP}(n_{ch} = 2) = P^{EXP}(n_- = 0) = P^{TH}(n = 0).$$

This is the case when the NBD is fitted. If instead the FNBD is fitted, the $P^{TH}(n = 0)$ and all probabilities for odd values of n are removed from the theoretical distribution. The $P^{EXP}(n_{ch} = 2)$ is fitted to the $P^{TH}(n = 2)$ (instead of $P^{TH}(n = 0)$). Hence the FNBD is shifted with respect to the NBD. Since in Ref. [10] the FNBD and not the NBD was fitted to the data, the obtained parameters k_{FNBD} are different from k_{NBD} obtained in this paper, which is shown in Table III.

Let us stress that for the FNBD formula (2) is no longer valid, that is

$$D^2_{FNBD} \neq \bar{n}_{FNBD} + \bar{n}^2_{FNBD}/k_{FNBD}. \tag{6}$$

Figure 1c illustrates the difference between the values of k fitted with the NBD (k_{NBD}) and FNBD (k_{FNBD}). The multiplicity tables for the UA5 data at $\sqrt{s} = 200$ and 900 GeV have not been published, so it was impossible to fit the NBD to the data, therefore instead of the k_{NBD} the k-value obtained by Method 1 (first column in Table III) has been used. The dramatic difference between k_{NBD} and k_{FNBD} at small energies is becoming negligible at the SPS Collider energy.

Note, that for $\sqrt{s} \approx 20$ GeV the value of k_{NBD} rises to infinity and at still lower energies becomes negative, whereas the value of k_{FNBD} for some of those data has still positive values (see Fig. 2a and Table III). This is because the FNBD is shifted with respect to the NBD. The authors of Ref. [10] fitted the FNBD instead of NBD and did not encounter problems connected with $k \to \infty$ only because data for $\sqrt{s} < 10$ GeV were discarded. Due to the shift the FNBD gives $k \to \infty$ and $k < 0$ at slightly lower energies. At very high energies the difference between NBD and FNBD decreases, since for broad enough multiplicity distributions both the shift and omission of every second bin in FNBD fits make relatively little change in the parameters $\bar{n}$ and k.

5. *k-parameter dependence on energy*

Parameter k is regarded to be essential for interpretation of the NBD fits within various models [1].

The k^{-1}_{NBD} is plotted in Fig. 2a as a function of $\sqrt{s}$ for the non-diffractive sample. The open points correspond to the fitted values of k_{NBD} (NBD fitted). The k_{FNBD} obtained in [10] are also presented (black points). The straight line corresponds to the relation $k^{-1} = -0.104 + 0.058 \ln s$ proposed in Ref. [11]. The line indeed represents a good fit to data for $10 < \sqrt{s} < 70$ GeV. However, the three points for $\sqrt{s} < 10$ GeV (omitted in [11]) deviate from the line towards smaller values of $1/k$, and the three points from the

UA5 experiment ($\sqrt{s} \geq 200$ GeV) are above the line. The open points representing fits to the true NBD show even less convincing linear dependence of k^{-1} with $\ln s$. Therefore we claim that the so often mentioned linear dependence of k^{-1} on $\ln s$ is not well proven. The check would be convincing with more accurate and numerous data, which are lacking. It seems to be worth trying to use the inelastic data to resolve the problem, having nevertheless in mind the difference between both samples of experimental data.

Figure 2b shows k_{NBD}^{-1} as a function of $\sqrt{s}$ for the inelastic data. Solid line corresponds to the relation (5) which is derived under the assumption that the NBD describes well multiplicity distributions and the data obey empirical linear dependence of the D vs $\langle n \rangle$ (Method 2). Rather than being linear in $\ln s$, the k^{-1} rises rapidly at small energies, passing $k^{-1} = 0$ at $\sqrt{s} \approx 10$ GeV, and then increases more slowly to the asymptotic value of A^2, where A is a constant in the relation (4). Note that taking into account the energy range covered by non-diffractive data used in [10, 11], that is for $\sqrt{s} > 10$ GeV, one could easily claim linear dependence of k^{-1} on $\ln s$ for the inelastic data.

In conclusion one can say that careful inspection of Figs. 2a and 2b does not support observation made by the authors of [10, 11] that k^{-1} is a linear function of $\ln s$. It was possible to obtain this type of relation because a few of the data points at the lowest energy were omitted and, generally speaking, because the non-diffractive data were too poor to provide hard enough constraints.

6. *Accidental KNO scaling*

In the UA5 Collaboration paper [10], the authors claim that the KNO scaling is accidentally valid in a certain energy range. The argument goes as follows. For the NBD the following relation holds:

$$C_{n,2} - 1 = \bar{n}^{-1} + k^{-1}, \tag{7}$$

where $C_{n,2} = \langle n^2 \rangle / \langle n \rangle^2$ and $n = 0, 1, 2 \ldots$

The sum $\bar{n}^{-1} + k^{-1}$ has a flat minimum in the energy range between 10 and 60 GeV (see Fig. 3a). Hence the $C_{n,2}$ moment is approximately constant in this energy range and the KNO scaling, which requires $C_{n_{\mathrm{ch}},2} = $ constant, is accidentally valid.

The fallacy of the argument is obvious. First, formula (7) is valid for the NBD, whereas the authors of [10] have fitted the FNBD and plotted $\bar{n}_{\mathrm{FNBD}}^{-1} + k_{\mathrm{FNBD}}^{-1}$ (Fig. 3a). But clearly

$$\bar{n}_{\mathrm{FNBD}}^{-1} + k_{\mathrm{FNBD}}^{-1} \neq \bar{n}_{\mathrm{NBD}}^{-1} + k_{\mathrm{NBD}}^{-1} = C_{n,2} - 1 \tag{8}$$

and, moreover

$$C_{n,2} \neq C_{n_{\mathrm{ch}},2}.$$

One can see that the energy dependence of $C_{n_{\mathrm{ch}},2}$ shown in Fig. 3b is different from that in Fig. 3a.

In addition simple arithmetic shows that

$$C_{n_{\mathrm{ch}},2} - 1 = \frac{\langle n_{\mathrm{ch}}^2 \rangle}{\langle n_{\mathrm{ch}} \rangle^2} - 1 = \frac{D_{\mathrm{ch}}^2}{\langle n_{\mathrm{ch}} \rangle^2}. \tag{9}$$

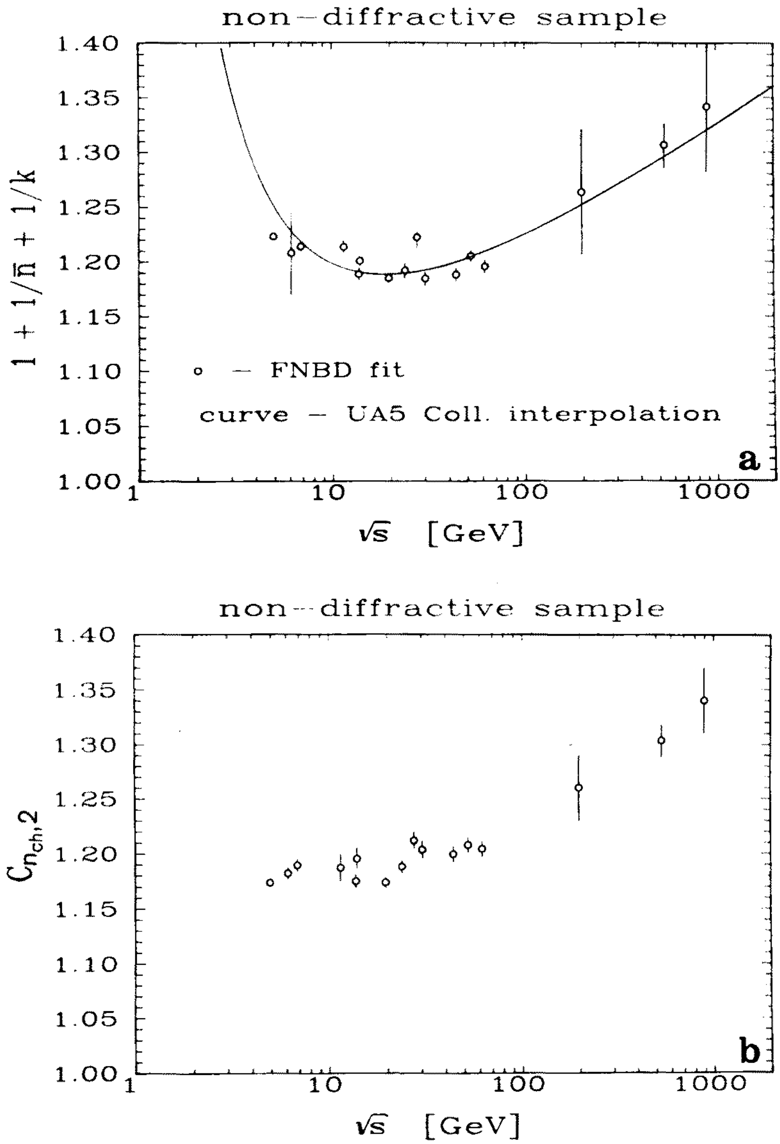


Fig. 3. a — The values of $1+1/\bar{n}+1/k$ from the FNBD fits to non-diffractive sample. The curve shows the UA5 Collaboration interpolation [11]. b — The values of $C_{n_{ch},2}$ moment for the same sample

At the threshold $\langle n_{ch}\rangle^2 = 4$ and $D_{ch} = 0$, hence $C_{n_{ch},2} = 1$ which means that the solid curve in Fig. 3a has to end up at the left bottom corner of the plot in complete disagreement with the dependence proposed in Ref. [10]. Thus the minimum seen in Fig. 3a has little to do with the KNO scaling.

Secondly, the constancy of the $C_{n_{ch},2}$ moment is not equivalent to the validity of KNO scaling [14]. The KNO scaling function ψ is continuous, whereas at non-asymptotic energies

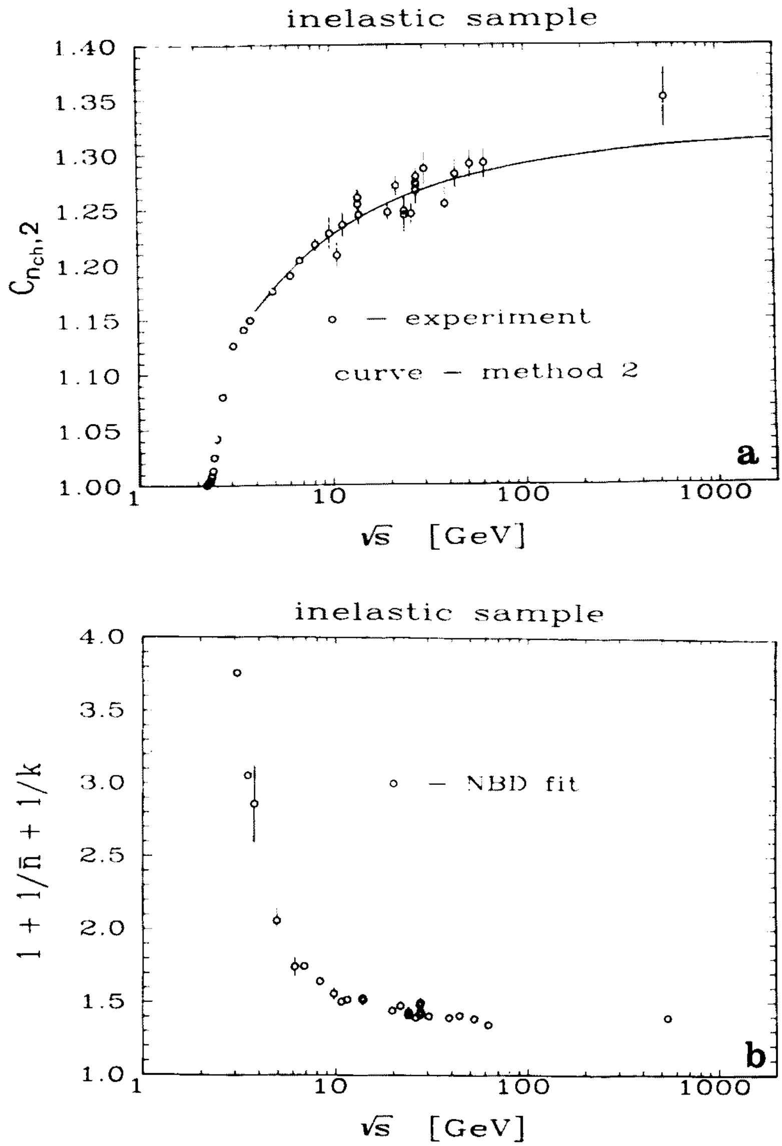


Fig. 4. a — The values of $C_{n_{ch},2}$ moment for the inelastic sample. The curve is obtained using relation (4) for total charged multiplicities. b — The values of $1+1/\bar{n}+1/k$ ($= C_{n,2}$) from the NBD fits to the same sample

we measure the discrete probability distribution P_n which has its moments different from those of ψ [15]. Thus, properly formulated KNO scaling, called KNO-G [14, 16], correctly describes the rise of $C_{n_{ch},2}$ moments with energy.

In our opinion the situation is clarified again by the inelastic data which are presented in Figs. 4a, b which show energy dependence of $C_{n_{ch},2}$ and $1+\bar{n}_{\rm NBD}^{-1}+k_{\rm NBD}^{-1}$, respectively.

Solid line in Fig. 4a corresponds to the relation (4). It is seen that the $C_{n_{ch},2}$ moments obtained directly from experimental data differ from $\bar{n}_{NBD}^{-1}+k_{NBD}^{-1}+1$. We remind that this is because $C_{n_{ch},2} \neq C_{n,2} = \bar{n}_{NBD}^{-1}+k_{NBD}^{-1}+1$. Note, that in the two plots there is no minimum in the energy dependence of plotted quantities contrary to the claim in Ref. [10].

The agreement of the data with the solid line is well understood in terms of the KNO-G scaling since relation (4) was derived just from the KNO-G scaling. Figure 4a illustrates the fact that the KNO-G scaling describes the data with great precision up to the top ISR energy [16]. However, the UA5 point at $\sqrt{s}$ = 540 GeV deviates from the curve indicating violation of the KNO-G scaling at this energy.

7. Is really Negative Binomial Distribution a new law in high energy collisions?

In the previous Section we have shown some typical misunderstandings and omissions in applying the NBD to the experimental data. In this Section we present further arguments in order to prove that the NBD can be treated only as a useful (but by no means the best) method of parametrizing the data on multiplicities.

Any formula pretending to the status of a law of physics has to (*i*) describe the experimental data very well, (*ii*) have a clear interpretation providing an insight into the physics mechanism of phenomena under study, and (*iii*) have a predictive power over hitherto unexplored regions of energy (and other relevant parameters). In what follows we attempt to show that the NBD does not meet the case.

Let us look again at the data collected in Table III. It can be seen that in most cases the χ^2 values for the NBD fits to the non-diffractive data are indeed reasonably small. However, the χ^2 value cannot serve as the only measure of the goodness of a fit. Equally important is the requirement that the data are scattered randomly around the fitted values. Fig. 5 shows clearly that it is not the case for the NBD fits. The values of the ratio $P^{EXP}(n_{ch})/P^{NBD}(n_{ch})$ (Fig. 5a) and $P^{EXP}(n_{ch})/P^{FNBD}(n_{ch})$ (Fig. 5b) are not scattered randomly around unity but systematic deviations can be seen. In other words, the NBD with its two parameters for each energy does not have enough flexibility to reproduce correctly the shape of multiplicity distributions.

At this point it is worth stressing an important difference between the NBD and KNO parametrizations of the data. In the KNO parametrization one arbitrary function is fixed for all energies, and there can be only one free parameter for all energies (see the method in Ref. [16]). In the NBD approach there are two free parameters for each energy. Assuming that $\langle n \rangle = \bar{n}$ this can be reduced to one free parameter, k, for each energy. It results in a considerable flexibility of the NBD which nevertheless, as shown above, is not large enough. In the KNO language this would mean a "scaling"(?!) function which may change with energy [17].

There have been attempts to fit multiplicity distributions with other formulas listed in Table I [4, 5, 18]. The fits were as good as those with the NBD which shows that the NBD is not unique as a way to parametrize multiplicity distributions.

There is another necessary condition for a distribution pretending to be "a law of

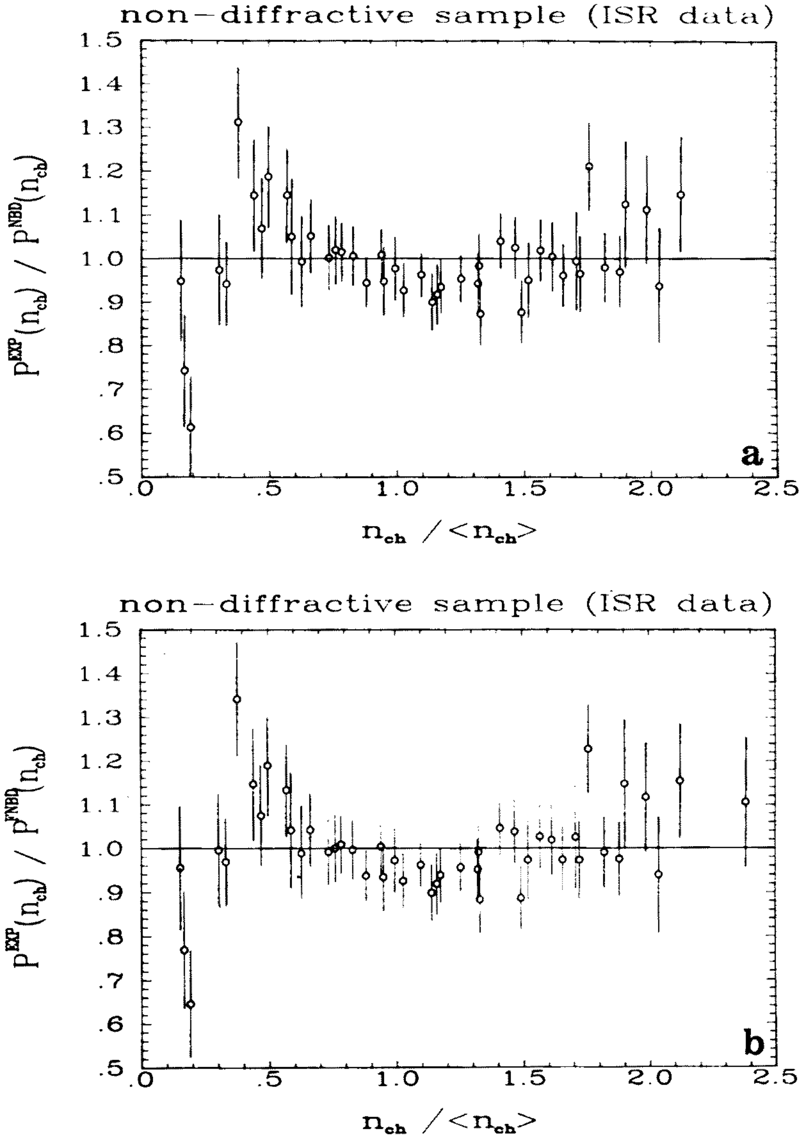


Fig. 5. a — The ratio of $P^{EXP}(n_{ch})/P^{NBD}(n_{ch})$ plotted as a function of reduced multiplicity $n_{ch}/\langle n_{ch}\rangle$ for the pp data at $30.4 < \sqrt{s} < 62.2$ GeV. b — The same for $P^{EXP}(n_{ch})/P^{FNBD}(n_{ch})$ ratio

physics", namely the stability of its parameters. We have checked the stability of parameters $\bar{n}$ and k by fitting the NBD to multiplicity distributions in which some data points have been omitted.

An example of the omission of some data points is shown in Fig. 6. The non-diffractive pp multiplicity distribution at 62 GeV contains 20 data points for $2 \leqslant n_{ch} \leqslant 40$. If five points at the low n_{ch} end are not used in the fit (Fig. 6b) one obtains considerably different

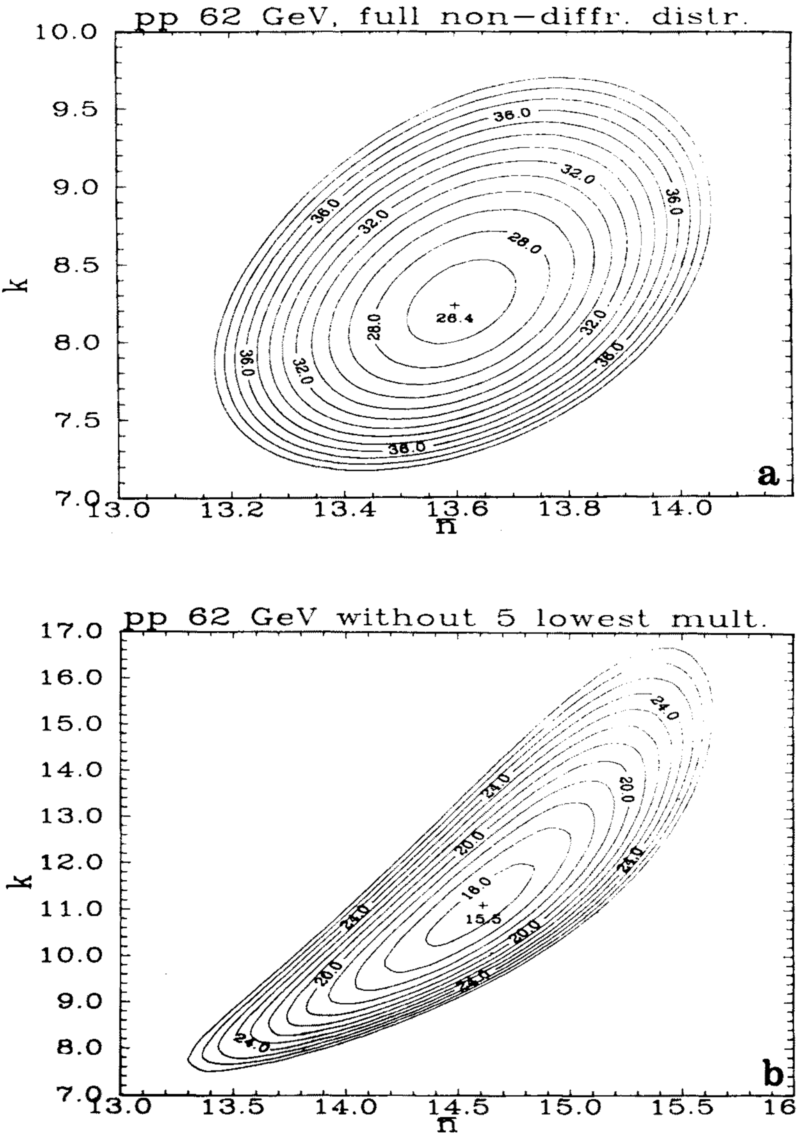


Fig. 6. The contours of constant χ^2 values in the FNBD fit to the non-diffractive pp multiplicity distribution at $\sqrt{s} = 62$ GeV as a function of parameters $\bar{n}$ and k; a — all data points used; b — data for $n_{\text{ch}} \leqslant 10$ omitted. Notice the shift in the best value of k between a and b

values of $\bar{n}$ and k than for the full distribution (Fig. 6a). Let us add here that the low n_{ch} end of multiplicity distribution is often omitted from the fit if inelastic data are used [19] because the FNBD is supposed to fit the non-diffractive sample and the diffractive component is known to be concentrated at $n_{\text{ch}} < 10$. If this supposition was true then one could estimate the diffractive component at each n_{ch} by subtracting the fitted cross sections from

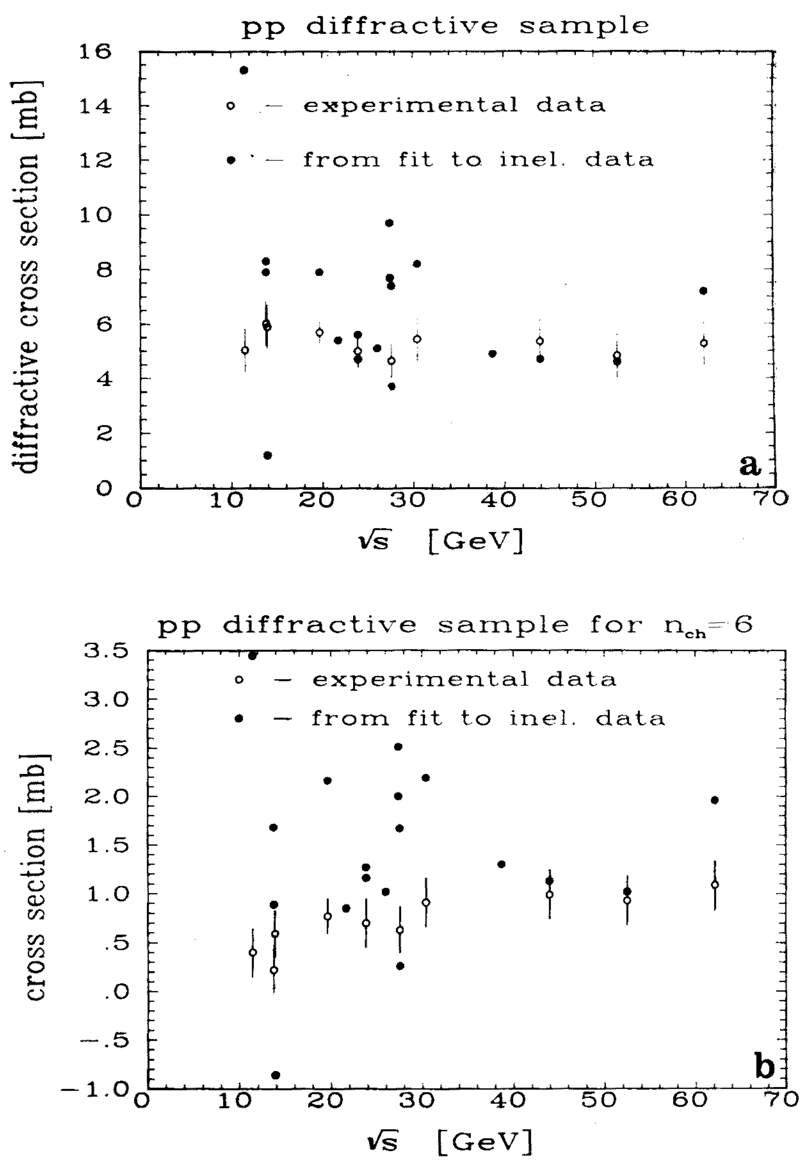


Fig. 7. Comparison of the experimental diffractive cross section with the "diffractive component" obtained by subtracting the NBD fitted cross sections from inelastic topologic cross sections for $n_{ch} \geqslant 10$; a — total cross section; b — diffractive cross section for $n_{ch} = 6$

inelastic topologic cross sections [19, 21]. Since we have started from non-diffractive distribution, the omission of a few lowest multiplicities should result in no change at all in the fitted parameters, assuming that indeed the described procedure of obtaining the non-diffractive data was justified. The difference of 30% in the fitted k_{FNBD} as seen in Fig. 6 makes the procedure questionable.

Moreover, as seen in Fig. 6b, the values of $\bar{n}$ and k are strongly correlated. This means that one gets approximately the same distribution by decreasing or increasing both $\bar{n}$ and k. This explains the sensitivity of the FNBD fits to the fluctuations in data and stresses again that the parameter k should not be considered as having any deep physical meaning.

Nevertheless, we have compared the diffractive cross sections obtained by other methods (missing mass plots, rapidity gaps etc.) with the "diffractive component" obtained by subtracting the NBD fitted cross sections from inelastic topologic cross sections. In Fig. 7a we see that the total diffractive cross sections from the NBD fits in many cases disagree with the experimental ones. There are clear systematic differences for separate topologies (Fig. 7b).

It is perhaps worth adding that for certain data the parameter k obtained from the fit has very large errors (see Table II and III) which again reminds that it has to be used with great care in evaluating some relevant physics parameters.

There are quite a few simple models in which the NBD can be derived [1]. We do not intend to characterize them (see Ref. [20] for a critical review) but we would like to point out that the k parameter is defined to be positive in all of them. As we have seen in Section 5, k^{EXP} at small enough energy becomes negative both in the NBD and FNBD fits. It is difficult to understand in terms of models why at a certain energy the k parameter should change sign although other experimental observables do not show any such violent change in the mechanism of particle production.

The other explanation is that the existing models have little connection with physics of multiparticle production.

Thus far we have discussed the NBD approach to full phase space non-diffractive and inelastic data. Much of what has been said above can also be repeated concerning the NBD fits to data in rapidity intervals, azimuthal angle intervals etc. In this connection we would like to point out that one should be careful in interpreting the difference in the values of k obtained in fits to distributions of negative secondaries (k_-) and all charged secondaries (k_{ch}). In the recent experiment the ratio k_-/k_{ch} for the different rapidity intervals was found to be two [21]. Fig. 1c shows however that for full phase space data the ratio k_-/k_{ch} is energy dependent and the difference between k_- and k_{ch} results from fitting different distributions: the NBD for negatives and the FNBD for all charged particles. It should be stressed that even for restricted phase-space regions the number of charged secondaries n_{ch} is not a good multiplicity measure because some charge correlations are still present. Again the number of negative secondaries serves as a genuine multiplicity measure for this case and it seems that only fits to the multiplicity distributions of negatives are meaningful.

Lastly, we would like to add that the NBD does not have any internal predictive power since it involves two free parameters for each multiplicity distributions and has no built-in scaling. There is no model to predict the dependence of $\bar{n}$ and k on the energy, particle type or a phase-space region in agreement with experiment. Therefore, extrapolation of parameters to higher energies is done by using the linear dependence of k on $\ln s$ claimed in [10, 11]. In Section 5 we have shown that this dependence is not well proven by the data. Also, there are no predictions for the variation of NBD parameters in different phase space regions. The usual conclusion, treated as a success of the NBD approach, is that its parameters change smoothly with the change of phase-space regions, but it would be rather difficult to imagine any other possibility.

Summarizing this Section we may say that there are no obvious reasons why the NBD should be particularly favoured as something more than a flexible mathematical parametrization of the results on multiplicities.

8. *Conclusions*

We have critically examined the existing evidence for the recent claim that the NBD is a new empirical law describing multiplicity of secondaries in high energy collisions. We have pointed out misunderstandings and incorrect procedure encountered in the literature on the subject. The shortcomings of the NBD are its insufficient flexibility to properly describe the shape of multiplicity distributions, lack of stability in the values of k, and the difficulty in the interpretation of the negative values of k in terms of models. We conclude that the NBD has not attained the status of a new physical law and should not be regarded as something more than a usable but not necessarily the best parametrization of the data. It is a mystery why it is so popular.

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