June 26, 1990

Warsaw University Preprint *IFD/5/1990*

Multiparticle production as a bivariate branching process

G. Wrochna

Institute of Experimental Physics

Warsaw University

PL-00-681 Warsaw, ul. Hoża 69

Abstract

Multiparticle production is studied as a branching process. Several approaches are compared. Especially, bivariate branching: in multiplicity and in energy is presented. In this approach the multiplicity distribution is found to be lognormal and obeying the KNO-G scaling. This result is very well confirmed by the experimental data.

Talk given at the XIII Warsaw Symposium on Elementary Particle Physics, Kazimierz, Poland, May 1990.

1 Introduction

The branching process is recently often postulated as a possible mechanism of the multiparticle production. Probably R.Hagedorn was the first who proposed this idea in 1965 to describe hadronic collisions [1]. However history of intensive studies on the subject starts from 1970 when Polyakov [2] has shown that the branching picture of the multiparticle production is implicated by the scale-invariant field theory with the similarity principle. It is interesting to note that he obtained the asymptotic scaling formula for multiplicities:

$$P_n = \frac{1}{\langle n \rangle} \psi \frac{n}{\langle n \rangle} . \tag{1.1}$$

– usually called KNO scaling – two years before Koba, Nielsen and Olesen [3].

In recent years QCD base of the branching was developed extensively: [4]–[6]. Presently several teams work on the subject: [7]–[18]. I collaborate with R.Szwed and A.K.Wróblewski and I would like to present our approach in comparison with others.

2 Branching models

Usually the following scenario of the multiparticle production is considered. Two primary quarks are produced in the e^+e^- annihilation. Each of them can emit a gluon which can convert into a new quark–antiquark pair. Each of the created quarks may again emit a gluon and so on.

Most of authors treat the evolution of the described cascade as a pure birth process. It means that in each vertex there is a certain probability $p(k)$ to produce k particles. It is schematically shown in Fig.1. The probability $p(k)$ is independent of the collision energy $\sqrt{s}$. The process is self-similar because the probability $p(k)$ is the same in each vertex. In the simplest case it is easy to find the final multiplicity distribution which is given by the so called Yule-Furry distribution:

$$P(n) = \frac{1}{\langle n \rangle} \left(1 - \frac{1}{\langle n \rangle} \right)^{n-1} . \tag{2.1}$$

The shape of this distribution is presented in Fig.2.

3 Difficulties

However, at this stage three difficulties appears.

1. The first concerns the shape of the distribution. The Yule-Furry distribution is a monotonously decreasing function in distinction from the multiplicity distributions measured in elementary particle collisions. To avoid the problem some authors propose to complicate the initial conditions and start the process from a certain number of particles k_0 rather than from a single vertex. Some authors assume that k_0 is fixed, others assume that the cascade phase is preceded by a coherent phase in which k_0 particles are produced with probabilities given by the Poisson distribution.

2. The second difficulty is the end of the cascade. In the pure birth process there is no mechanism to stop the particle production. Such a mechanism has to be "put by hand" into the model. In other words, the probability $p(k)$ in "last" vertices should be changed to enable an extinction of the cascade. The process is no longer self-similar.

The natural point to cut the cascade is when the energy of the parton is of the order of pion mass. This leads us to the third problem:

3. How to take into account the energy. Some authors proposes to identify the energy with the "time" parameter.

4 Our approach

In our approach the energy of partons in the cascade is treated as a second variable, having the same rank as a multiplicity. It is illustrated in Fig.3. The thickness of a branch represents the energy of a given parton. Now, in each vertex there is a bivariate probability $p(k, E)$ to produce k particles with energy E. This resolves all the three difficulties at once. Parton energy is taken into account. The cascade can be terminated without violation of self–similarity, because we can put (in each vertex, of course) $p(k, E \leq m_\pi) \to 0$. The shape of the resulting distribution is different from the Yule-Furry distribution.

5 Distribution of gold grains

To calculate the shape of the multiplicity distribution in this approach we need only a bit of imagination. Let me try to answer the question, closely related to the parton multiplicities: what is the probability to find a gold grain of a given mass? (Fig.4.)

Some experimental data on the subject was published in the Soviet gold industry journal "Sovietskaja Zolotopromyshlennost" in 1937 [19]. Unfortunately, I have not found this volume, instead the distribution for copper grains [20] is shown in Fig.5., which – from a mathematical point of view – is completely the same. We see in Fig.5. that the logarithm of mass is distributed normally. In other words, mass is distributed lognormally. Kolmogorov, inspired by the presented data, assumed that gold grains are created by a crushing process [21]. Let us start from a gold grain of mass m_0. The grain can break and create k pieces of mass m_1 with probability $p(k, m_1)$. Each piece can break again and so on. Now we see why the question about gold grains is related to the elementary particle collisions.

6 Derivation of the lognormal distribution

It is easy to see that the considered process has a multiplicative nature, i.e. the number of particles (or gold grains) in a given generation is proportional to the number of particles in the previous one.

$$n_i = (1 + \epsilon_i)\, n_{i-1} \ . \tag{6.1}$$

In other words, the final number of particles is a product of a number of random factors:

$$n = n_0 \prod_i (1 + \epsilon_i) \ . \tag{6.2}$$

Taking logarithms of both sides and expanding $\ln(1 + \epsilon_i)$ near 1 we see that the right hand side of equation (6.2) is a sum of a number of random components:

$$\ln(n/n_0) = \sum \epsilon_i \ . \tag{6.3}$$

Such a sum should be distributed normally due to the Central Limit Theorem. Hence, also a left hand side, i.e. a logarithm of multiplicity should be distributed normally. In other words, multiplicity should be distributed lognormally.

$$P(n) = \frac{N}{\sqrt{2\pi}\sigma} \cdot \frac{1}{n} \exp\left(-\frac{[\ln n - \mu]^2}{2\,\sigma^2}\right). \tag{6.4}$$

7 Continuous multiplicity

However, one should be careful to compare this result with experimental data at finite energies. Formally we get the lognormal distribution in the limit of a large number of steps. Thus, the lognormal distribution is continuous. For very large n it is not important, but for finite energies we should be careful to apply continuous distribution for discrete multiplicities. To explain how to do it, let me use a Galton board as an example.

As you know, Galton board is a device to "produce" the normal distribution. It consists with a number of pins, arranged as shown in Fig.6. and several slots at the bottom. When small balls are pushed through the system of pins, they fall into the slots and create kind of a histogram. If the number of slots is large, the histogram content can be well approximated by the value of the normal distribution at a given point. However, when the number of slots is rather small and the size of a single slot is not negligible, we should integrate the normal density over the range occupied by a single slot.

The same procedure can be applied in the case of the lognormal distribution.

$$P_n = \int_n^{n+1} P(n)\, dn\ . \tag{7.1}$$

8 KNO–G scaling

In a similar way we can write the scaling formula (called KNO-G [22,23]) valid at finite energies in distinction from the asymptotic formula of Koba, Nielsen and Olesen [3].

$$P_n = \int_n^{n+1} \frac{1}{\langle\tilde{n}\rangle} \psi\left(\frac{n}{\langle\tilde{n}\rangle}\right) dn = \int_{n/\langle\tilde{n}\rangle}^{(n+1)/\langle\tilde{n}\rangle} \psi(z) dz\ . \tag{8.1}$$

where $\langle\tilde{n}\rangle = \int_0^\infty n\psi(z)dz$ is an average continuous multiplicity [22,23].

The lognormal scaling function $\psi(z)$ has now the following form:

$$\psi(z) = \frac{N}{\sqrt{2\pi}\sigma} \cdot \frac{1}{z + c'} \exp\left(-\frac{[\ln(z + c') - \mu']^2}{2\,\sigma^2}\right). \tag{8.2}$$

More details can be found in Ref.[24,25].

9 Comparison with experiment

Now let me show you the proof that multiplicity distributions in e^+e^- collisions are indeed lognormal. To enable a graphical test we transform the coordinates system in such a way that lognormal curve is transformed into a straight line. The scheme of the transformation in sketched in Fig.7. First we use the reduced multiplicity z to unify distributions measured at various energies. Then, we take the logarithm of the horizontal axis to transform the lognormal distribution to the normal one. Next, we take the sum of P_n to obtain a distribution function. Finally, we transform the vertical axis according to the inverted normal distribution function. If we have got a straight line after all these transformations then the distribution is lognormal. This kind of plot is called the probit diagram [26]. It is a very sensitive test because it is easy to distinguish any curve from a straight line.

Fig.8. shows the probit diagram for the e^+e^- multiplicities. In this figure 25 data sets from various experiments are collected [27]. The initial energy varies between 3 and 44 GeV. One can see that points follow a straight line, i.e. the multiplicities are distributed lognormally.

Moreover, this picture proves the scaling. In principle, each set of data could have followed a separate straight line. The fact that they follow a single line means that the data obey the scaling.

Another graphical test is presented in Fig.9a. Experimentally measured probability $P_n(exp)$ is divided by the probability calculated for the lognormal distribution $P_n(lognormal)$. The ratio $P_n(exp)/P_n(lognormal)$ is plotted as a function of the reduced multiplicity z. Data points scatter randomly around unity, which means that there are no systematic deviations of the data from the lognormal curve.

10 The lognormal and the negative binomial distribution

It is interesting to compare the lognormal distribution with the negative binomial distribution. The ratio of measured and calculated probabilities for the negative binomial distribution $P_n(exp)/P_n(NBD)$ is plotted in Fig.9b. In distinction from Fig.9a systematic deviations are seen.

I would like to point out the χ^2/NDF is also much better for the lognormal distribution (0.68) than for the NBD (1.5) in spite of the fact that the parameters of the lognormal distribution are fixed for all energies, whereas the negative binomial distribution requires two free parameters to be fitted at each energy.

11 Conclusions

Now, let me go to the conclusions. We have calculated the multiplicity distribution in e^+e^- collisions assuming a kind of branching process as a mechanism of the multiparticle production. We have obtained the lognormal distribution with the scaling property. We have checked that the data confirm this result very well.

The Central Limit Theorem enabled us to get the final result without any assumption about a functional form of the single vertex probability $p(k, E)$. We assumed only that $p(k, E)$ is independent of the collision energy $\sqrt{s}$ (scale–invariant) and that it is the same for each vertex (self–similarity). In other words, the final multiplicity distribution does not contain any information about dynamics in a single vertex. In that sense the QCD can not be tested by studying multiplicities. However the information carried by the multiplicity distributions was sufficient to reconstruct the general mechanism of the multiparticle production, that it is a branching, self–similar and scale–invariant process.

Obviously every model based on the QCD should have these properties. And in this sense multiplicity distributions can test the QCD. We can use the knowledge about the branching process to create more realistic models and to make predictions for the future experiments.

References

[1] R.Hagedorn: Nuovo Cimento Suppl. 3 (1965) 147.

[2] A.M.Polyakov: Sov. Phys. JETP 32 (1971) 296; 33 (1971) 850 (in Russian: Zh. Eksp. Teor. Fiz. 59 (1970) 542; 60 (1971) 296).

[3] Z.Koba, H.B.Nielsen and P.Olesen: Nucl. Phys. B40 (1972) 317.

[4] K.Konishi, A.Ukawa, G.Veneziano: Nucl. Phys. B157 (1979) 45.

[5] A.Basetto, M.Ciafaloni, G.Marchesini: Nucl. Phys. B163 (1980) 477.

[6] G.Marchesini and B.R.Webber: Nucl. Phys. B310 (1988) 461.

[7] M.Biyajima, N.Suzuki: Phys. Lett. 143B (1984) 463; Progress of Theor. Phys. 73 (1985) 918; Phys. Lett. B189 (1987) 466; Ann. Inst. Statist. Math. 40 (1988) 229.

[8] C.C.Shih: Phys. Rev. D33 (1986) 3391;
P.Carruthers, C.C.Shih: Int. Jour. of Mod. Phys. A2 (1987) 1447.

[9] P.V.Chliapnikov, O.G.Chikilev: Phys. Lett. B222 (1989) 152; Phys. Lett. B223 (1989) 119; Serpukhov preprint IFVE-89-201, submitted to Pisma Zh. Eksp. Teor. Fiz.; Phys. Lett. B235 (1990) 347.

[10] P.Cvitanovič, P.Hoyer, K.Konishi: Phys. Lett. 85B (1979) 413;
P.Cvitanovič, P.Hoyer, K.Zalewski: Nucl. Phys. B176 (1980) 429.

[11] N.K.Dushutkin, V.M.Maltsev: Dubna preprint JINR-P2-89-375.

[12] M.Anselmino, A.Giovannini, G.Vercellino: Nuovo Cim. 62A (1981) 253;
A.Giovannini: Nucl. Phys. B161 (1979) 429;
G.Vercellino: Lett. Nuovo Cim. 32 (1981) 369.

[13] M.I.Gorenshtein, G.M.Zinovev, V.K.Petrov, Yu.M.Sinyukov: Pisma Zh. Eksp. Teor. Fiz. 32 (1980) 47.

[14] C.S.Lam: 21st Recontre de Moriond (1986) Hadronic v.2, 241;
R.C.Hwa, C.S.Lam: Mod. Phys. Lett. A1 (1986) 303; Phys. Rev. D34 (1986) 3370;
W.R.Chen, R.C.Hwa: Phys. Rev. D36 (1987) 760; Phys. Rev. D39 (1989) 179;
R.C.Hwa: Phys. Rev. D37 (1988) 1830; Oregon preprint OITS-374; Oregon preprint OITS-380, 2nd Int. Conf. on Elastic and Diffractive Scattering, New York (1987), 2nd Blois, 299; Oregon preprint OITS-384, published in Shandong Workshop (1987) 328; Nucl. Phys. B328 (1989) 59;
R.C.Hwa, X.Wang: Oregon preprint OITS-397, published in Arles Multipart. Dyn. (1988) 377; Oregon preprint OITS-404, to be published in "Multiparticle Production", Proc. of Perugia Workshop, Perugia, Italy, Jun 21-28, 1988; Oregon preprint OITS-410;
C.B.Chiu, R.C.Hwa: Phys. Lett. B236 (1990) 466; Oregon preprint OITS-431.

[15] S.J.Orfanidis, V.Rittenberg: Phys. Rev. D10 (1974) 2892.

[16] A.D'Innocenzo, P.Rotelli: Nuovo Cim. Lett. 35 (1982) 65;
P.Rotelli: Nuovo Cim. 80A (1984) 205;
G.Mussardo, P.Rotelli: Mod. Phys. Lett. A2 (1987) 847.

[17] S.Sakai: Phys. Rev. D40 (1989) 1430.

[18] I.Sarcevic: Minnesota Univ. preprint UMN-TH-86-2704, "Multiparticle Dynamics", Seewinkel, Austria (1986) 639; Minnesota Univ. preprint UMN-TH-86-554; Santa Fe Strong Int. Workshop (1986) 160; Mod. Phys. Lett. A2 (1987) 513;
B.Durand, I.Sarcevic: Phys. Lett. B172 (1986) 104; Phys. Rev. D36 (1987) 2693;
I.Sarcevic, H.Satz: Phys. Lett. B233 (1989) 251.

[19] Sovietskaja Zolotopromyshlennost 12 (1937) 39.

[20] N.K.Razumovsky: Comptes Rendus de l'Academie des Sciences de l'URSS (Doklady Akademii Nauk SSSR) 28 (1940) 814.

[21] A.N.Kolmogorov: Comptes Rendus de l'Academie des Sciences de l'URSS (Doklady Akademii Nauk SSSR) 31 (1941) 99.

[22] A.I.Golokhvastov: Sov. J. Nucl. Phys. 27 (1978) 430; 30 (1979) 128.

[23] R.Szwed and G.Wrochna: Z. Phys. C – Particles and Fields 29 (1985) 255.

[24] R.Szwed and G.Wrochna: Warsaw University Preprint IFD/3/1989, to be published in Z. Phys. C – Particles and Fields (1990).

[25] R.Szwed, G.Wrochna and A.K.Wróblewski: Warsaw University Preprint IFD/1/1990, to be published in Modern Physics Letters (1990).

[26] N.Arley and K.R.Buch: Introduction to the Theory of Probability and Statistics, New York 1950, Chapter 10, (John Wiley and Sons).

[27] *MARK I* (3.0, 4.0, 4.8, 7.4 GeV) J.L.Siegrist et al.: Phys. Rev. D26, (1982) 969
LENA (7.4, 8.9, 9.3 GeV) B.Niczyporuk et al.: Z. Phys. C – Particles and Fields 9 (1981) 1
JADE (12, 30, 35 GeV) W.Bartel et al.: Z. Phys. C – Particles and Fields 20 (1983) 187
HRS (29 GeV) M.Derrick et al.: Phys. Rev. D34 (1986) 3304
PLUTO (3.6, 5.0, 7.7, 9.4, 12.0, 13.0, 17.0, 22.0, 27.5, 30.6 GeV) private communication
TASSO (14.0, 22.0, 34.8, 43.6 GeV) W.Braunschweig et al.: Z. Phys. C – Particles and Fields 45 (1989) 1939.

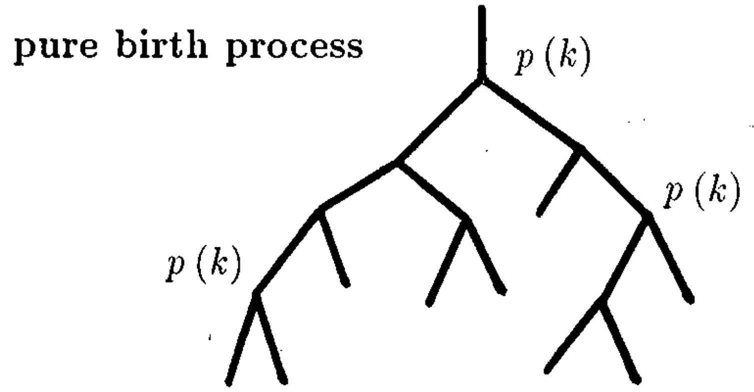


Fig.1

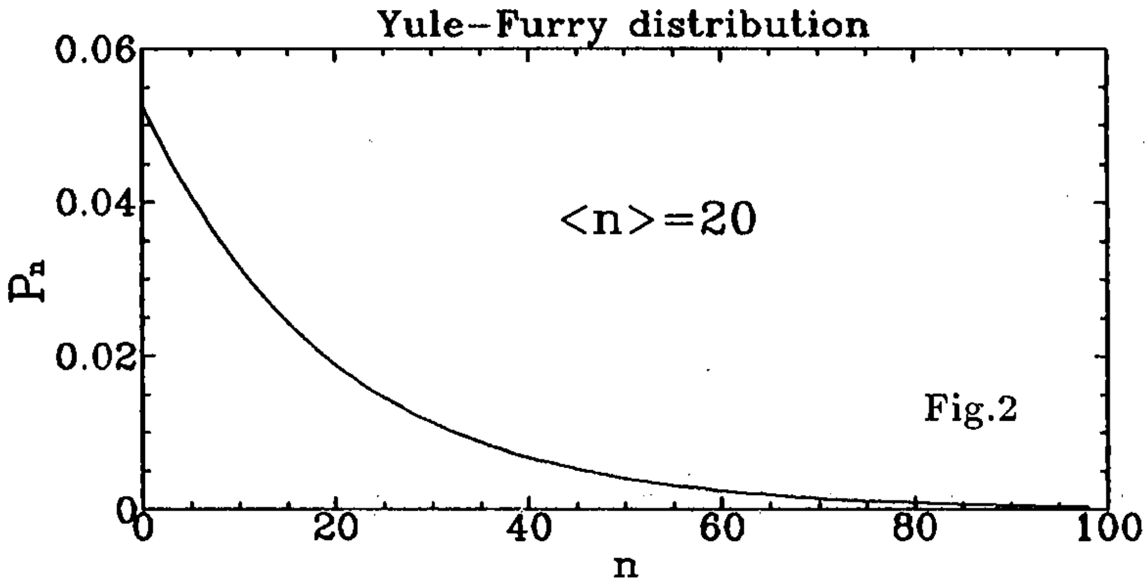


Fig.2

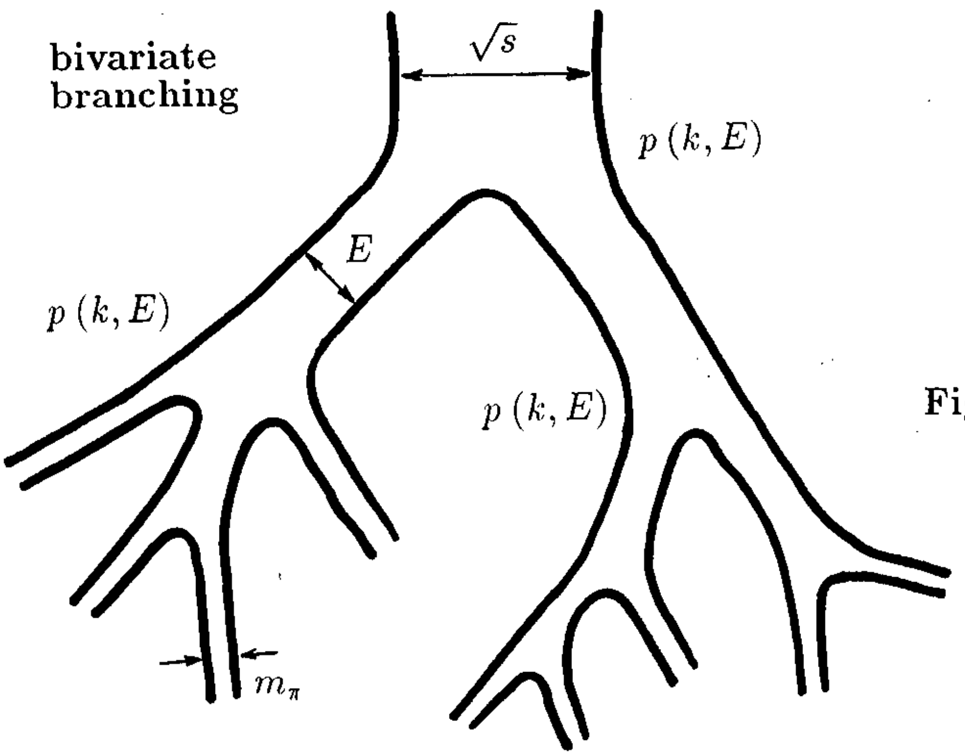


Fig.3

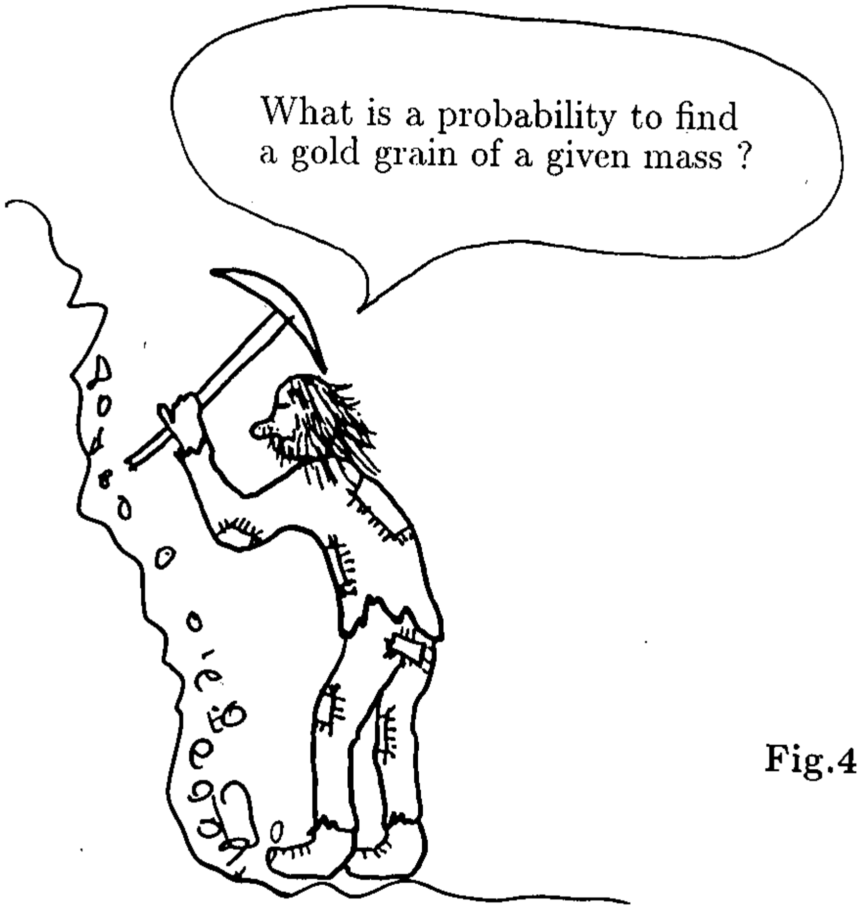

What is a probability to find
a gold grain of a given mass ?

Fig.4

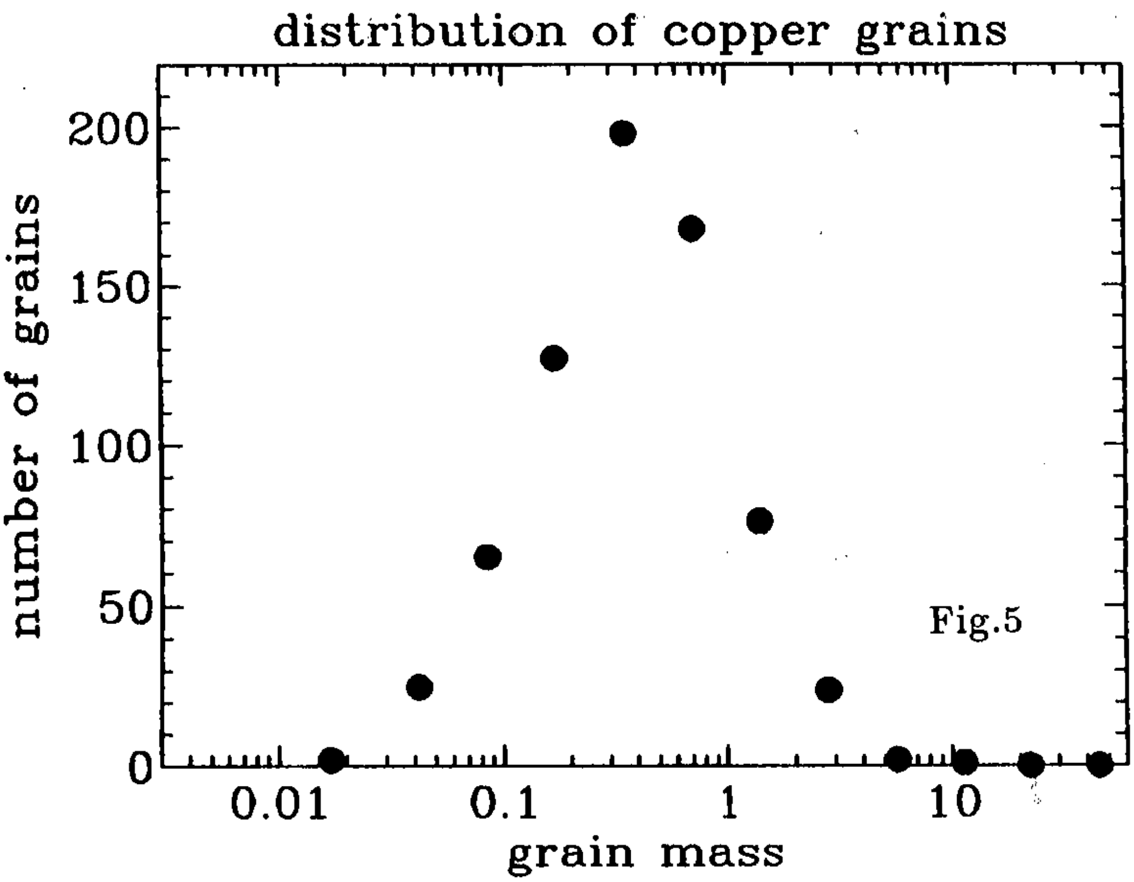

distribution of copper grains
number of grains
0
50
100
150
200
0.01
0.1
1
10
grain mass

Fig.5

GALTON BOARD

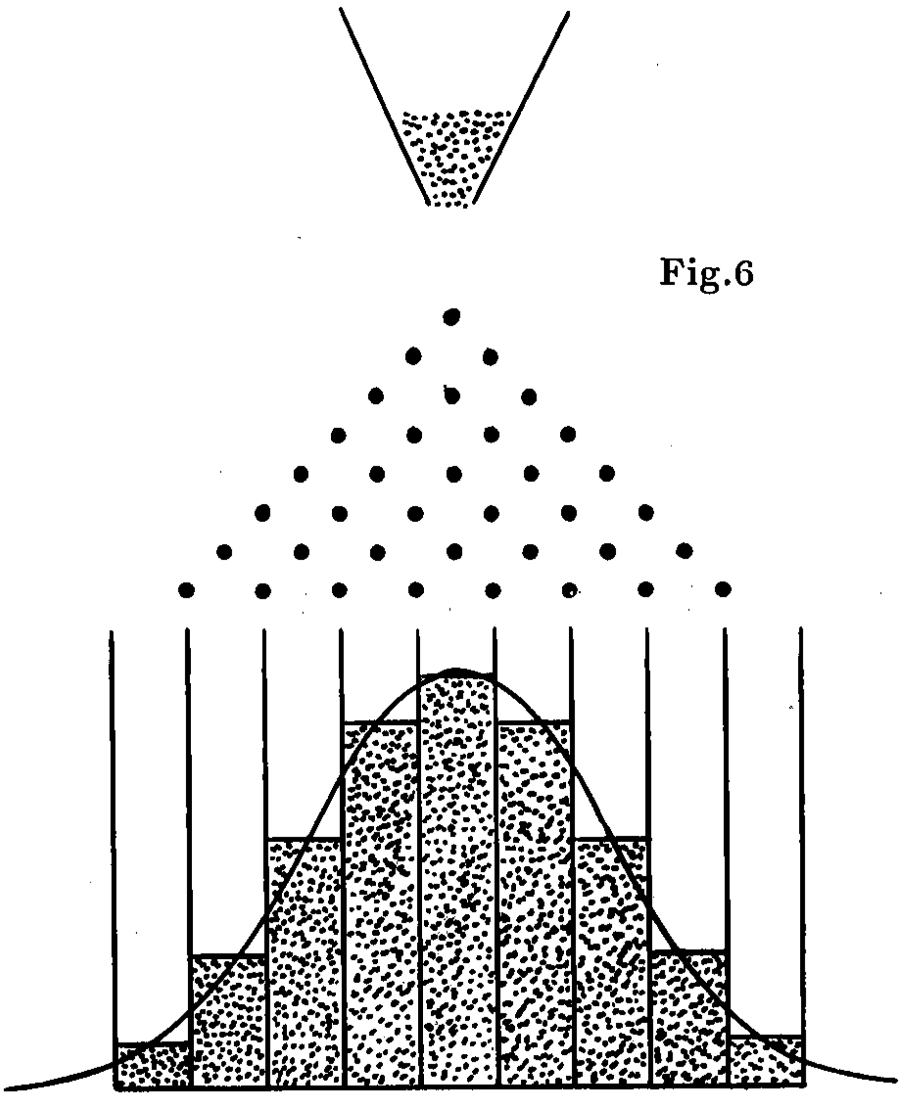
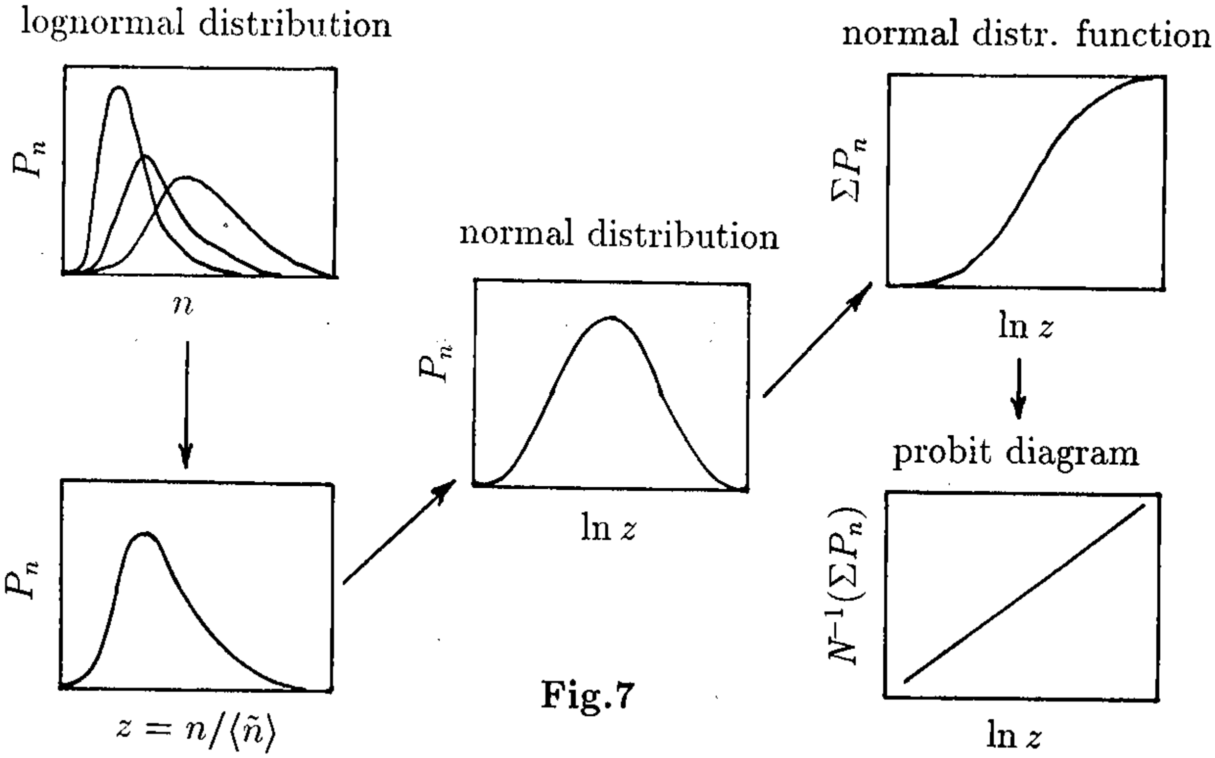


Fig.6

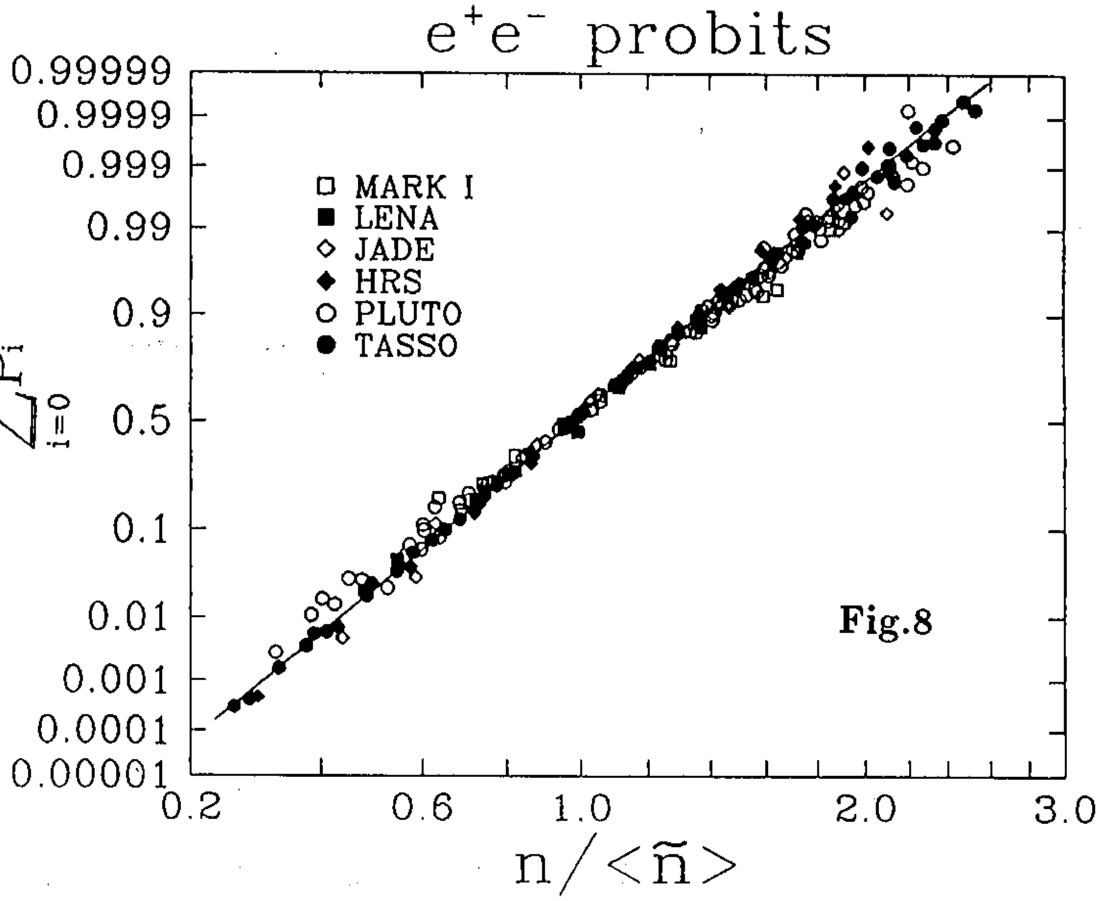
large n: $\quad P_n \approx P(n)$

better: $\quad P_n = \int_n^{n+1} P(n)\, dn$



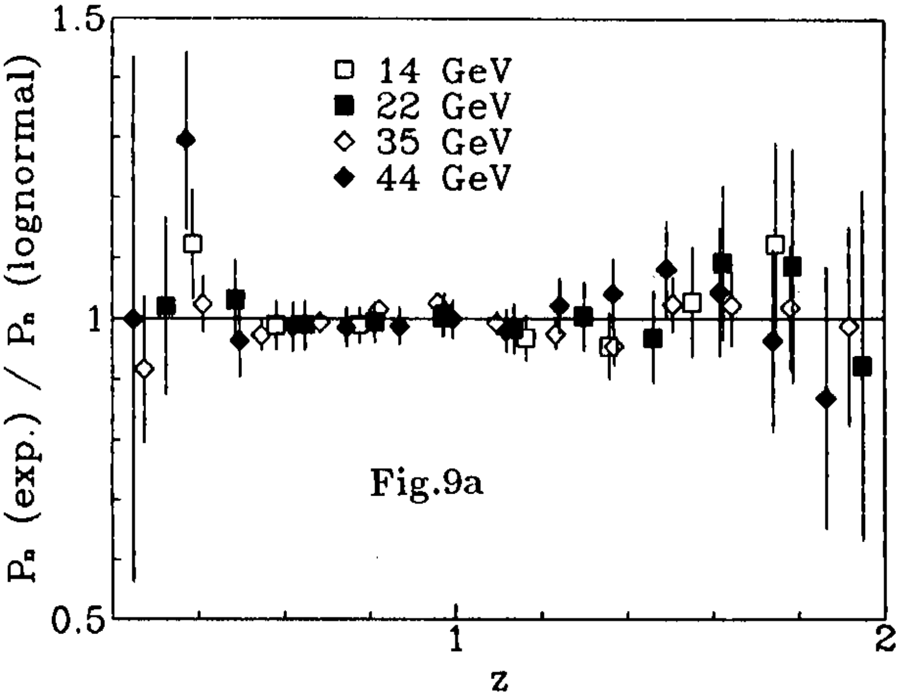
lognormal distribution
P_n
n
P_n
$z = n/\langle\tilde{n}\rangle$
normal distribution
P_n
$\ln z$
normal distr. function
ΣP_n
$\ln z$
probit diagram
$N^{-1}(\Sigma P_n)$
$\ln z$

Fig.7

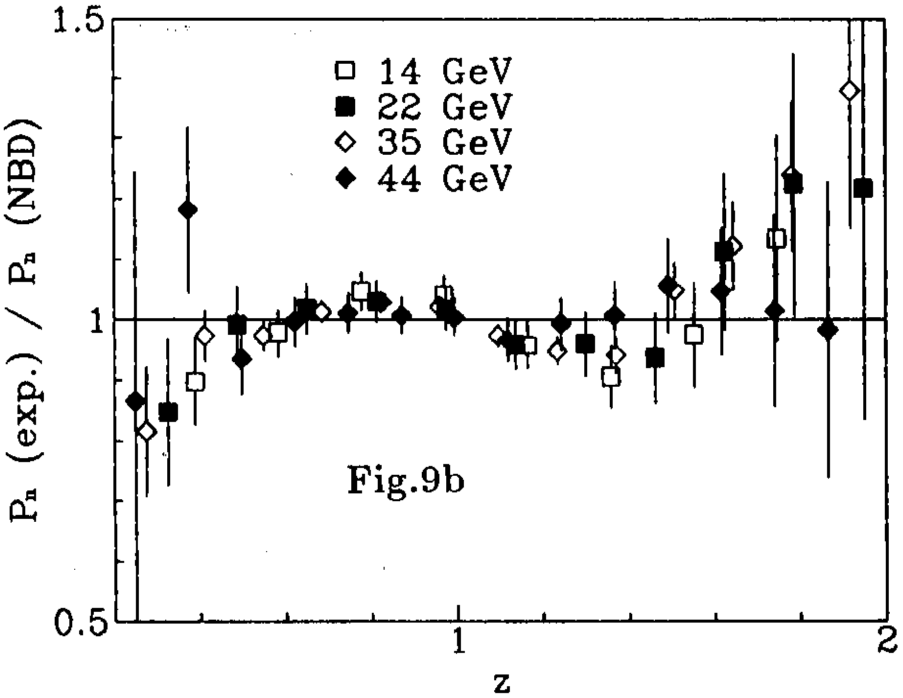


e⁺e⁻ probits
0.99999
0.9999
0.999
0.99
0.9
0.5
0.1
0.01
0.001
0.0001
0.00001
MARK I
LENA
JADE
HRS
PLUTO
TASSO
$\sum_{i=0}^{n} P_i$
0.2
0.6
1.0
2.0
3.0
$n/\langle\tilde{n}\rangle$

Fig.8



14 GeV
22 GeV
35 GeV
44 GeV
Fig.9a
1.5
1
0.5
1
2
z
Pn (exp.) / Pn (lognormal)



14 GeV
22 GeV
35 GeV
44 GeV
Fig.9b
1.5
1
0.5
1
2
z
Pn (exp.) / Pn (NBD)